Étale cohomology #2

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Basics on morphisms of schemes

Finiteness assumptions

Affine morphisms

Separated and proper morphisms

Finite, quasi-finite and unramified morphisms

Finite and quasi-finite morphisms (defn's and ex's)

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Examples

Unramified morphisms

Summary

Finiteness assumptions

Milne: all rings noetherian
all schemes locally noetherian, i.e.

$$X = \underbrace{\bigcup U_i}_{\text{open covering}} \quad U_i = \operatorname{Spec}(A_i) \quad A_i \text{ noetherian}$$

Def: X quasi-compact if every open covering has a finite subcovering

$$\Leftrightarrow X = \bigcup_{i=1}^{n} U_i, \ U_i \text{ affine}$$

Def: X noetherian if locally noetherian + quasi-compact

Def: $f: X \to Y$ (morphism of schemes) is locally of finite type if $\forall$ comm. diagrams

$$\begin{tikzcd} X \arrow[d, "f"'] & U = \operatorname{Spec}(B) \arrow[l, "\text{open imm}"'] \arrow[d, "g"] \arrow[dl, phantom, "\circ"] \\ Y & V = \operatorname{Spec}(A) \arrow[l, "\text{open imm}"] \end{tikzcd}$$

B f.gen A-algebra.

(enough to cover X with such diagrams)

Def: f quasi-compact if $\forall U \subset Y$ affine (or merely quasi-compact) $f^{-1}(U)$ is quasi-compact.

Def. f finite type if loc. fin. type + quasi-compact.

Ex: finite type over $\operatorname{Spec}(k)$ $\Rightarrow$ noeth.
loc ——— " ——— $\Rightarrow$ loc noeth.

X/k quasi-projective, i.e. $X \subset \mathbb{P}^n_k$ loc closed $\Rightarrow$ X finite type $/k$.

Rmk: We can often assume that our schemes are of fin type over a field but we also need $\operatorname{Spec} \mathcal{O}_{X,x}$, $\operatorname{Spec} \mathcal{O}^h_{X,x}$, $\operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ which are not of finite type.

Affine morphisms

Def: $f: X \to Y$ affine if $\exists\, Y = \bigcup U_i$ open affine covering s.th. $f^{-1}(U_i)$ affine $\forall i$.

Sorités: (i) stable under basechange:

$$\begin{tikzcd} X & X' = X \times_Y Y' \arrow[l] \arrow[d, "f'"] \\ Y \arrow[from=u, "f"'] \arrow[ur, phantom, "\square"] & Y' \arrow[l, "g"] \end{tikzcd}$$

f affine $\Rightarrow$ f' affine

(ii) stable under compositions: f, g affine $\Rightarrow$ $f \circ g$ affine

(iii) closed immersions are affine.

Relative Spec: Y scheme, $\mathcal{A}$ quasi-coherent sheaf of $\mathcal{O}_Y$-algebras

$$X := \underline{\mathrm{Spec}}_Y(\mathcal{A}) \xrightarrow{f} Y \qquad \text{locally} \qquad f^{-1}(U) = \mathrm{Spec}\, \mathcal{A}(U) \to U = \mathrm{Spec}\, R \qquad \text{corr to} \qquad R = \mathcal{O}_Y(U) \to \mathcal{A}(U)$$

f is affine. Can recover $\mathcal{A} = f_* \mathcal{O}_X$.

Fact: Bijection $\{ f: X \to Y \text{ affine} \} \longleftrightarrow \{ \mathcal{A} \text{ qcoh sheaf of } \mathcal{O}_Y\text{-alg} \}$

$$f \longmapsto f_* \mathcal{O}_X$$

$$\underline{\mathrm{Spec}}_Y(\mathcal{A}) \longmapsfrom \mathcal{A}$$

Rmk: f affine $\iff$ $\forall$

$$\begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[l] \arrow[d] \\ Y & Y' = \mathrm{Spec}\, A \arrow[l] \end{tikzcd}$$

X' is affine.

Fact: Given
$$\begin{tikzcd} X \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd}$$
, if g affine then (and f qcqs)

$$\mathrm{Hom}_Y(X, Z) = \mathrm{Hom}_{\mathcal{O}_Y\text{-alg}}(g_*\mathcal{O}_Z, f_*\mathcal{O}_X)$$

In particular, $\exists$ canonical factorization "Stein factorization"

$$f : X \xrightarrow{\alpha} \underline{\mathrm{Spec}}_Y\, f_*\mathcal{O}_X \xrightarrow{\beta} Y$$

f is affine $\Leftrightarrow$ α isomorphism.

Aside: quasi-affine morphisms

Def: $f : X \longrightarrow Y$ quasi-affine if $\exists$ $X \xrightarrow[\text{open immersion}]{} \overline{X} \xrightarrow[\text{affine}]{} Y$

Fact: f quasi-affine $\Leftrightarrow$ α is an open immersion.

Separated and proper morphisms

Def: $f: X \to Y$ separated if $\Delta_f : X \to X \times_Y X$ closed immersion.

Rmk: f separated $\iff \exists\, Y = \bigcup U_i$ affine open covering s.th. $X_i = f^{-1}(U_i)$ separated

that is $\Delta_{X_i} : X_i \to X_i \times X_i$ closed immersion

Rmk: separated analogous to Hausdorff for topological spaces.

$X/\mathbb{C}$, X separated $\iff X^{an}$ Hausdorff.

Def: $f: X \to Y$ proper if

(i) f separated
(ii) f locally of finite type
(iii) f universally closed

$$\begin{tikzcd} \forall\ X \arrow[d, "f"'] & X' \arrow[l] \arrow[d, "f'"] \arrow[dl, phantom, "\square"] \\ Y & Y' \arrow[l] \end{tikzcd}$$

f' is closed: $f'(E) \subset Y'$ closed $\forall E \subset X'$ closed.

Rmk: proper analogous to proper for top. spaces (= closed w/ compact fibres)

$X/\mathbb{C}$, X proper $\iff X^{an}$ compact

$f: X \to Y$ $/\mathbb{C}$, f proper $\iff f^{an}$ proper

Ex: Quasi-projective $\Rightarrow$ separated (and varieties are by def^n separated)
Projective $\Rightarrow$ proper
Affine $\Rightarrow$ separated

Sorités: separated, proper stable under base change and composition.

closed immersion $\Rightarrow$ proper

(indeed: an immersion is closed iff it's proper)

Finite and quasi-finite morphisms (Milne I.1)

Def: $f: X \to Y$ is

(i) finite if affine + $f_* \mathcal{O}_X$ finitely generated as $\mathcal{O}_Y$-module.

(ii) quasi-finite if finite type + discrete fibers

i.e. $f^{-1}(y)$ discrete ($\Leftrightarrow$ finite set of closed and open points)

Sorites (Milne I.1.3, I.1.7)

- finite, quasi-finite stable under base change and composition.
- immersions are quasi-finite
- closed immersions are finite.

Rmk: $f: Z \hookrightarrow X$ closed immersion. Locally given by

$\mathrm{Spec}(A/I) \hookrightarrow \mathrm{Spec}\, A \quad \leftrightarrow \quad A/I \twoheadleftarrow A$.

A/I is always a fin. gen. A-module (generator 1.)

Rmk: $f: X \to Y$ finite. Locally given by

$\mathrm{Spec}(A[x_1, \ldots, x_n]/I) \to \mathrm{Spec}\, A \quad \leftrightarrow \quad A[x_1, \ldots, x_n]/I \leftarrow A$

s.th. every x_i satisfies a monic equation: (i.e. $A \to A[x]/I$ integral ext.)

$$x_i^n + a_{n-1} x_i^{n-1} + \cdots + a_1 x_i + a_0 = 0 \qquad a_j \in A$$

If A alg. closed field, then $\exists$ finite number of solutions for each x_i

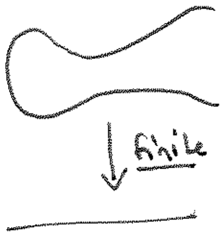
$\Rightarrow$ fibers of f are finite, i.e. f quasi-finite.

Normalization: X integral scheme (locally $\mathrm{Spec}\, A$ where A integral domain + connected)

Then $\exists$ normalization $\tilde{X} \xrightarrow{\pi} X$. It is integral, i.e. affine and $\mathcal{O}_X \to \pi_* \mathcal{O}_{\tilde{X}}$ integral extension.

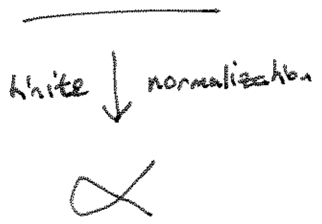
π finite if X finite type / field.

Ex:



elliptic curve $\qquad$ locally $\qquad$ $\mathbb{P}^1$

$$\begin{array}{c} k[x,y]/y^2 - x(x-1)(x-\lambda) \\ \uparrow \text{finite.} \\ k[x] \end{array}$$

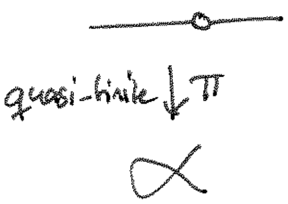


$$\begin{array}{c} A[z]/z^2 - x - 1 \cong k[z] \\ \uparrow \text{finite} \\ A = k[x,y]/y^2 - x^3 - x^2 \end{array}$$

" $z = y/x$ "

$x = z^2 - 1$

$y = z^3 - z$



all fibers of π consists of exactly 1 point (π bijective)

not finite.

Spec $A_f \longrightarrow$ Spec A open immersion $\Rightarrow$ quasi-finite.

typically not finite

Exercise (Milne I.1.6)

(a) Let A DVR, $P(T) \in A[T]$. Let $B = A[T]/(P(T))$. Show that Spec $B \longrightarrow$ Spec A is

- quasi-finite $\iff$ some coefficient of $P(T)$ is a unit
- finite $\iff$ leading —— " ——

(b) Let A Dedekind domain, fraction field K. Show that Spec $K \to$ Spec A is

- never finite
- quasi-finite $\iff$ finite type $\iff$ Spec A finite set.

Finite vs quasi-finite

Thm (Milne I.1.10, EGA III 4.4.2, IV 18.12.4)

$f : X \to Y$. TFAE

(i) f finite
(ii) f affine + proper
(iii) f quasi-finite + proper

pf: (i) $\Rightarrow$ (ii): f affine by definition $\Rightarrow$ separated. also of finite type by def. finite $\Rightarrow$ univ closed (Cohen-Seidenberg's "going up") } proper

(ii) $\Rightarrow$ (iii): Can assume $Y = \operatorname{Spec} k$. Then finiteness thm for proper morphisms $\Rightarrow f_* \mathcal{O}_X$ finite k-vector space $\Rightarrow f$ quasi-finite.

(iii) $\Rightarrow$ (i): Stein factorization $X \xrightarrow{\alpha} X' = \underline{\operatorname{Spec}}_Y f_* \mathcal{O}_X \xrightarrow{\beta} Y$
[assuming Y noetherian]
f proper $\Rightarrow$
- α connected non-empty fibers + proper
- β finite.

f quasi-finite $\Rightarrow$ α bijective

Thus α homeomorphism. Since $\alpha_* \mathcal{O}_X = \mathcal{O}_{X'}$ it follows that α isom. $\square$

Zariski's main thm (Milne I.1.8, EGA III 4.4.3, IV 8.12.6, 18.12.13)

$f: X \to Y$. TFAE (assume Y qcqs)

(i) f quasi-finite and separated

(ii) $\exists$ factorization $f: X \xrightarrow[\text{open immersion}]{\alpha} \bar{X} \xrightarrow[\text{finite}]{\beta} Y$

Rmk: ZMT implies (iii) $\Rightarrow$ (i) of previous thm.
Indeed: f proper $\Rightarrow$ α proper $\Rightarrow$ α closed and open immersion $\Rightarrow$ α finite $\Rightarrow$ f finite.

Sketch of pf of ZMT when f quasi-projective:

Since f q-proj: $\exists\, X \xrightarrow[\text{open imm.}]{g} X' \xrightarrow[\text{projective}]{h} Y$.

Let $U \subseteq X'$ open subset of X' where h is quasi-finite. Note $X \subseteq U$.

Take Stein factorization of h: $X' \xrightarrow[\text{proper conn fibers}]{} X'' \xrightarrow[\text{finite}]{} Y$.

Then $h(U)$ open and $U \xrightarrow{\cong} h(U)$ isomorphism
(first prove homeomorphism, then iso b/c $\mathcal{O}_{X''} \xrightarrow{\cong} h_*\mathcal{O}_{X'}$)

$$f: \underbrace{X \subset U \xrightarrow{\cong} h(U) \subset X''}_{\text{open immersion}} \xrightarrow[\text{finite}]{} Y$$

□

General case: Use complete local rings / henselian local rings.

Exercise (Milne I.1.12)

$f: X \to Y$ separated, finite type, Y irreducible w/ generic point η.

Show that $f^{-1}(\eta)$ finite $\Rightarrow$ $\exists$ non-empty open $U \subset Y$ s.th. $f^{-1}(U) \to U$ finite.

Universally injective morphisms

Def: $f: X \to Y$ **universally injective** (radicial) if

$$\forall \quad \begin{tikzcd} X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X' \arrow[l] \arrow[d, "f'"] \\ Y & Y' \arrow[l] \end{tikzcd} \qquad f' \text{ is injective.}$$

Exercise: $f: X \to Y$. TFAE

(i) f univ. inj.

(ii) Δ_f surjective

(iii) $\forall y \in |Y|$: either $f^{-1}(y) = \emptyset$ or $f^{-1}(y) = \{x\}$ and $\kappa(y) \to \kappa(x)$ purely inseparable extension.

(iv) $\forall$ fields K, $\mathrm{Hom}(\mathrm{Spec}\, K, X) \longrightarrow \mathrm{Hom}(\mathrm{Spec}\, K, Y)$ is injective.

Rmk: univ inj + finite type $\Rightarrow$ quasi-finite.

Monomorphisms

Def: $f: X \to Y$ monomorphism if $\forall T$, $T \overset{g_1}{\underset{g_2}{\rightrightarrows}} X$: $f\circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2$

that is, $\mathrm{Hom}(T, X) \to \mathrm{Hom}(T, Y)$ injective.

Rmk: mono $\Rightarrow$ univ inj.

Lemma: $f: X \to Y$ mono $\iff$ $\Delta_f : X \to X \times_Y X$ iso.

pf: Two maps $T \overset{g_1}{\underset{g_2}{\rightrightarrows}} X$ such that $f\circ g_1 = f\circ g_2$ gives rise to $T \xrightarrow[(g_1,g_2)]{} X \times_Y X$. We have diagram:

$$\begin{tikzcd}
 & X \arrow[d, "\Delta_f"] \arrow[dr, "id"] & & \\
T \arrow[ur, dashed, "g"] \arrow[r, "{(g_1,g_2)}"'] & X \times_Y X \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] & X \arrow[r, "f"] & Y
\end{tikzcd}$$

There's a lift $g: T \to X$ of (g_1, g_2) iff $g_1 = g_2$.

Thus Δ_f iso $\Rightarrow$ f mono.

Conversely, $f\circ\pi_1 = f\circ \pi_2$ so if f mono then $\pi_1 = \pi_2$.

It follows that $\Delta\circ\pi_i = id$ (look at $\pi_j(\Delta\circ\pi_i) = \pi_j$)

Always: $\pi_i \circ \Delta = id$. Thus π_i inverse to Δ. $\square$

Prop (EGA IV 17.2.6) $f: X \to Y$ morphism of schemes. Consider

(i) f mono

(ii) $\forall y \in |Y|$: either $f^{-1}(y) = \emptyset$ or $f^{-1}(y) \to \mathrm{Spec}\, \kappa(y)$ isomorphism.

Then (i) $\Rightarrow$ (ii) and converse holds if f locally of finite type.

pf: (i) $\Rightarrow$ (ii) reduces to $Y = \mathrm{Spec}\, k$. Then fairly easy.

(ii) $\Rightarrow$ (i): uses notion of <u>unramified</u>!

Ex: $f : \operatorname{Spec}(K) \to \operatorname{Spec}(k)$, $k \hookrightarrow K$ field extension

f quasi-finite $\iff$ K/k finite

f univ injective $\iff$ K/k purely inseparable

f monomorphism $\iff$ $K = k$

f unramified $\iff$ K/k separable

Ex: $\operatorname{Spec}(k[\varepsilon]/\varepsilon^2) \to \operatorname{Spec} k$ univ injective, not mono nor unramified

Ex: 

$= \mathbb{A}^1 \setminus \{1\} = \operatorname{Spec} k[z]_{z-1}$ $\qquad z = y/x \qquad x = z^2 - 1$, $y = z^3 - z$

$f \downarrow$

 $=$ nodal cubic $= \operatorname{Spec} k[x,y]/(y^2 - x^3 - x^2)$

f monomorphism : $f^{-1}(\text{origin}) \cong \operatorname{Spec} k$ (exercise)

and bijective but not finite

Ex: $\qquad = \mathbb{A}^1 = \operatorname{Spec} k[z]$ $\qquad z = y/x \qquad x = z^2$, $y = z^3$

$f \downarrow$ normalization

 $=$ cuspidal cubic $= \operatorname{Spec} k[x,y]/(y^2 - x^3)$

f finite, univ. inj, surj but not mono : $f^{-1}(\text{origin}) \cong \operatorname{Spec}(k[z]/(z^2))$

not unramified

Unramified morphisms (Milne I.3)

Prop./Def (Milne I.3.1) A finite k-algebra is <u>separable</u> if the following <u>equivalent</u> conditions hold.

$$A \cong \prod_{i=1}^{n} k_i \quad k_i/k \text{ separable}$$

$$\Leftrightarrow A \otimes_k \bar{k} \cong \prod_{i=1}^{m} \bar{k} \quad \Leftrightarrow A \otimes_k \bar{k} \text{ is reduced (i.e. nilradical} = 0)$$

$\Leftrightarrow$ discriminant of any basis of A is non-zero.

Prop/def (Milne I.3.5) $f : X \longrightarrow Y$ is <u>unramified</u> if f locally of finite type and TFEQ hold:

(i) $\forall y \in |Y|$, $f^{-1}(y) = \operatorname{Spec} A$ where A finite separable $\kappa(y)$-algebra

(ii) Δ_f is an open immersion

(iii) $\Omega^1_{X/Y} = 0$ ($\Leftrightarrow f$ is formally unramified)

<u>Sketch of pf</u>: (i) $\Rightarrow$ (iii): Enough to prove $\Omega^1_{X/Y}|_{f^{-1}(y)} = 0$ (Nakayama)

$$\Omega^1_{X/Y}|_{f^{-1}(y)} = \Omega^1_{f^{-1}(y)/\kappa(y)} = \widetilde{\Omega^1_{A/k}} = 0$$

$\uparrow$ cond (i)

(iii) $\Rightarrow$ (ii) Let $I \subset \mathcal{O}_{X \times_Y X}$ ideal defining $\Delta_f : X \overset{\text{loc closed}}{\hookrightarrow} X \times_Y X$. in a <u>nbhd</u> U of X.

U open, I closed U

Then $\Omega^1_{X/Y} = I/I^2$ (as an $\mathcal{O}_X$-module)

$\Omega^1_{X/Y} = 0 \Rightarrow I = 0$ in a nbhd of X (Nakayama)

$\Rightarrow X \hookrightarrow U$ open

(ii) $\Rightarrow$ (i). WLOG $Y = \operatorname{Spec} \bar{k}$. Then every closed point x of X is a $\bar{k}$-point. $X \xrightarrow{f} Y = \operatorname{Spec} \bar{k}$, with section x. Δ_f open $\Rightarrow x$ open

so $X = \coprod \operatorname{Spec} \bar{k}$. $\square$

Rmk: unramified $\Rightarrow$ loc quasi-finite (using (i))

Sections of unramified morphisms are open (using (ii))

— " — sep + unram — " — clopen

monomorphism $\Rightarrow$ unramified (use (i), (ii) or (iii)!)

Lemma (EGA IV 17.2.6) $f: X \to Y$ loc of finite type. TFAE

(i) f monomorphism

(ii) f unramified + universally injective

pf: (i) $\Rightarrow$ (ii): easy; use fiberwise condition.

(ii) $\Rightarrow$ (i): (ii) $\Leftrightarrow$ Δ_f open + Δ_f surj $\Leftrightarrow$ Δ_f iso $\Leftrightarrow$ (i) $\square$

Proves postponed fact that f mono $\Leftrightarrow$ $f^{-1}(y)$ empty or isom. to Spec $\kappa(y)$.

(cf Milne I.3.12)

Prop: $f: X \to Y$ unramified.

$$\left\{ \begin{array}{l} \text{sections of } f, \text{ i.e.} \\ s: Y \to X \text{ s.th.} \\ \quad f \circ s = id_X \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{l} \Gamma \overset{\text{open}}{\subset} X \text{ s.th.} \\ \Gamma \to X \xrightarrow{\pi_2} Y \text{ iso.} \end{array} \right\}$$

$$f \longmapsto f(Y)$$

If f also separated, then Γ clopen.

If in addition Y connected, then sections of f $\longleftrightarrow$ $\left\{ \begin{array}{l} \text{conn. comp of } X \\ \text{mapping iso to } Y \end{array} \right\}$

and any section determined by its value at one point.

Summary

$$\begin{tikzcd}
 & \text{finite} \arrow[r, Leftrightarrow, "\text{ZMT}"] \arrow[d, Rightarrow] & \text{quasi-finite} + \text{proper} \\
 & \text{quasi-finite} & \\
\text{universally injective} \arrow[ur, Rightarrow] & & \text{unramified} \arrow[ul, Rightarrow] \\
 & \text{monomorphism} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] \arrow[d, Leftrightarrow] & \\
 & \text{univ inj} + \text{unram.} &
\end{tikzcd}$$

fiberwise characterization

categorical & diagonal characterizations

$$\begin{tikzcd}
\Delta\ \text{surjective} & & \Delta\ \text{open imm} \\
 & \Delta\ \text{isomorphism} \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] &
\end{tikzcd}$$