Étale cohomology #14

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Recall: comparison

~~Finiteness~~

Recall $H^i(X_{\acute{e}t}, M) \cong H^i(X(\mathbb{C}), M)$

X (smooth) var / $\mathbb{C}$

M finite abelian grp (or locally constant finite, or constructible)

If X connected: $H^1(X_{\acute{e}t}, M) \cong \check{H}^1(X_{\acute{e}t}, M) \cong PHS(M/X) \cong \mathrm{Hom}_{cont}(\pi_1(X,\bar{x}), M)$

If X also normal: $H^1(X_{\acute{e}t}, \mathbb{Z}) \cong \mathrm{Hom}_{cont}(\pi_1(X,\bar{x}), \mathbb{Z}) = 0$ b/c π_1 pro-finite so image finite.

but $H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^{2g}$ when X sm proj curve of genus g.

Similarly, $H^2(\mathbb{P}^1_{\mathbb{C},\acute{e}t}, \mathbb{Z}) = 0$ and not $\mathbb{Z}$ as "expected".

Finiteness

Def: F sheaf on $X_{\acute{e}t}$.

- F finite if $F(U)$ finite $\forall U \to X$ ét.
- F has finite stalks if $F_{\bar{x}}$ finite $\forall \bar{x} \to X$.
- F locally constant if $\exists (U_i \to X)$ covering s.th. $F|_{U_i}$ const

Recall: X connected. Then F loc. const w/ finite stalks $\overset{1-1}{\Longleftrightarrow}$ finite $\pi_1(X,\bar{x})$-sets

$F \longmapsto F_{\bar{x}}$

$\updownarrow$ 1-1

$h_Z \longleftarrow\!\shortmid (Z \to X)$ $\quad$ FEt_X

so for any X, loc. const w/ finite stalks

$\Updownarrow$

finite Étale / X

Rmk: $\pi: Y \to X$ proper, F loc const on Y

Then $\pi_* F$ not always loc const but constructible.

In fact even $R^i\pi_* F$ constructible if F constructible

(intimately conn. to proper base change)

Espace étalé

Thm: Any sheaf F (of sets or abelian grps) on X_{Et} is repr. by a unique (loc. sep.) algebraic space $\tilde{F}$ étale over X

$$\left(\pi: X_{Et} \to X_{et} \text{ gives } \pi^*: Sh(X_{et}) \xrightarrow[\cong]{\text{equiv of cat}} \{\text{alg. sp. étale over } X\} \subset Sh(X_{Et}) \right)$$

with inverse π_*.

Alg. sp. /X: $R \underset{i_2}{\overset{i_1}{\rightrightarrows}} U$, R, U schemes$/X$, is an equiv relation

if $\forall T \to X$, $R(T) \rightrightarrows U(T)$ equiv relation of sets

A sheaf of sets on the big étale site X_{Et} is an alg sp$/X$ if

$\exists$ equiv relation $R \underset{i_2}{\overset{i_1}{\rightrightarrows}} U$ where i_1, i_2 étale (and surj autom.)

(and $R \xrightarrow[(i_1,i_2)]{} U \times_X U$ q-cpt if we restrict to q-sep alg. sp.)

s.th. $F = U/R$ is the associated sheaf to $T \mapsto U(T)/\sim$.

Def.: F loc. separated if $R \to U \times_X U$ an immersion. $\quad$ ($\Leftrightarrow F \xrightarrow{\Delta_F} F \times_X F$ imm)

$\quad$ separated $\quad$ — " — closed immersion. ($\Leftrightarrow$ — " — closed imm)

Equiv def of alg sp: F sheaf on X_{Et} s.th.

(1) $\exists U \to X$ étale, surj, representable, U a scheme

(2) $\Delta_F: F \to F \times_X F$ representable $\qquad$ Rmk: (1) $\Rightarrow$ (2) and (2) $\Rightarrow U \to X$ repr

Def: F normal, étale$/X$, loc. of fin. type$/X$, ...

$\Leftrightarrow U$ ——— " ———

(makes sense for any étale-local property)

Ex: If G finite grp acting on Y scheme, then Y/G does not need to be a scheme but is an alg. space and $Y \to Y/G$ finite étale.

Ex: A group alg. space of finite type over a field is a group scheme (Artin)

Constructible sheaves

Def: $F \in Sh(X_{ét})$ is constructible if $\tilde{F} \to X$ of finite type.

(Milne V.1.8)
Prop: TFAE for $F \in Sh(X_{ét})$

- F constructible
- $\forall Z \subset X$ (irr.) closed, $\exists$ open dense $U \subset Z$ s.th. $F|_U$ loc. const + finite stalks.
- $X = \bigcup U$ (finite if X qc.) where $U \subset X$ loc closed and $F|_U$ ——— " ———

l-adic sheaves

$Sh(X_{ét}, \mathbb{Z}/n\mathbb{Z})$ (coh as $\mathbb{Z}/n\mathbb{Z}$-mod same as Ab groups = $\mathbb{Z}$-mod.)

From now: loc. const assumes finite stalks.

Dual: $\check{F} := \mathcal{H}om(F, \mathbb{Z}/n)$

X scheme, res. char. prime to n $\rightsquigarrow$ μ_n étale sheaf, loc free sheaf of $\mathbb{Z}/n\mathbb{Z}$-modules of rank 1.

Twisting

$$\mathbb{Z}/n\mathbb{Z}(r) := \begin{cases} \mu_n \otimes \dots \otimes \mu_n & r > 0 \\ \mathbb{Z}/n\mathbb{Z} & r = 0 \\ \mathbb{Z}/n\mathbb{Z}(-r)^\vee & r < 0 \end{cases}$$ (Tate twist)

$F(r) := F \otimes \mathbb{Z}/n(r)$ Tate twist.

ℓ prime number.

ℓ-adic sheaf on $X_{ét}$: proj system $F = (F_n)$, F_n sheaf on $X_{ét}$

s.th $F_{n+1} \longrightarrow F_n$ induces $F_{n+1}/\ell^n F_{n+1} \xrightarrow{\sim} F_n$.

($\Rightarrow F_n$ $\mathbb{Z}/\ell^n\mathbb{Z}$-module)

ℓ-adic cohomology

$$H^r(X, F) := \varprojlim H^r(X, F_n) \quad \left(\neq H^r(X, \varprojlim_n F_n)\right)$$

$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z}$ acts on $H^r(X, F)$

Def: F loc "const (twisted const) / constr if F_n loc const / constr. $\forall n$.

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Def: F lisse

Rmk: F twisted constant ($=$ lisse) $\not\Rightarrow$ loc const in the sense that $\exists\, U \longrightarrow X$ fin. étale surj