Étale cohomology #13

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Torsors

$G \longrightarrow X$ flat + l.f.pres group scheme $\quad \left(G \times_X G \xrightarrow{\mu} G, \quad X \xrightarrow{e} G \quad \text{etc} \right)$

An (right) <u>action</u> of G on an X-scheme E is a morphism

$$E \times_X G \xrightarrow{\rho} E$$

s.th. ①

$$\begin{tikzcd} E \times_X G \times_X G \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\rho \times \mathrm{id}"'] \arrow[dr, phantom, "\circ"] & E \times_X G \arrow[d, "\rho"] \\ E \times_X G \arrow[r, "\rho"] & E \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "\mathrm{id} \times e"] \arrow[rr, bend right, "\mathrm{id}"'] & E \times_X G \arrow[r, "\rho"] \arrow[d, phantom, "\circ"] & E \\ & {} & \end{tikzcd}$$

or <u>equivalently</u> ② $\forall$ X-schemes T

$$E(T) \times G(T) \xrightarrow{\rho(T)} E(T)$$

is an action of $G(T)$ on $E(T)$.

<u>Ex</u>: G acts on itself by right multiplication $(\rho = \mu)$ or conjugation.

$$\left(\rho(x, g) = x^g = g^{-1} x g \right)$$

An action gives rise to an equivalence relation:

$$E \times_X G \xrightarrow{\psi} E \times_X E$$

$$(e, g) \longmapsto (e, eg)$$

ψ mono $\iff$ action is <u>free</u>

ψ surj $\iff$ action is transitive (pointwise)

ψ iso $\iff$ action is free + transitive (in strong sense)

$G(T)$ acts freely + transitively on $E(T)$ $\quad \forall T \to X$

<u>Def</u>: $E \to X$ is a <u>G-torsor</u> if $E \times_X G \xrightarrow{\varphi} E \times_X E$ <u>iso</u>.
and $E \to X$ faithfully flat and loc. of fin. pres.

(also called <u>principal homogenuous space</u>)
or principal bundle

<u>Rmk</u>: A torsor is like a group <u>w/o unit</u>. Indeed, given any section $s : X \to E$ of $E \to X$, we obtain: G-equivariant map

$$G \xrightarrow{s \times id} E \times_X G \xrightarrow{\rho} E$$

which is an iso since $G(T) \xrightarrow{\sim} E(T)$ $\forall T \to X$.
$g \longmapsto sg$

<u>Locally</u>, a torsor has a (non-canonical) unit:

<u>Prop</u> (III 4.1): $E \to X$ is a G-torsor iff

(*) $\exists$ a covering $(U_i \to X)_i$ for the flat topology s.th.

$$E|_{U_i} \cong G|_{U_i} \quad G|_{U_i}\text{-equivariantly.}$$

<u>pf</u>: If $E \to X$ is a G-torsor, take $U := E$. Then

$$E|_U = \cancel{E} \times_X \cancel{E} = E \times_X E \xrightarrow[\varphi^{-1}]{\sim} E \times_X G = G|_U \quad G|_U\text{-equivariantly}$$

(alternative point of view: the torsor $E \times_X E \to E$ has a section Δ hence is isomorphic to $E \times_X G$.)

Conversely if (*) holds, then by descent theory

$$E \times_X G \xrightarrow{\varphi} E \times_X E \text{ iso} \iff E \times_X G \times_X U_i \xrightarrow{\varphi_{U_i}} E \times_X E \times_X U_i \text{ iso } \forall i$$

$$E \longrightarrow X \text{ fflat, lfp} \iff E \times_X U_i \longrightarrow U_i \text{ f.flat, l.f.p } \forall i$$

and RHS holds by (*).

<u>Def</u>: A G-torsor E is <u>trivial</u> if $\exists$ a section. Then $E \cong G$ non-canonical.

(any section $s \in E(X)$ gives an iso $E \to G$).

<u>Def</u>: $PHS(G/X) = \{ E \to X : G\text{-torsor} \} / \text{iso}$. a <u>pointed</u> set.

(A morphism of torsors is a G-equivariant map $E_1 \to E_2$.
Any such map is an iso.)

Distinguished element: the trivial G-torsor $G \to X$.

Prop says that torsors are locally trivial.

<u>Rmk</u>: By descent, if G smooth/étale, then any G-torsor is smooth/étale.

Also, if G smooth, then G-torsors are étale-locally trivial.

<u>Ex</u>: $H \hookrightarrow G$ subgroup (G, H flat $/X$).

Suppose G/H is a scheme (always an alg space).

Then $G \to G/H$ is an H-torsor.

More generally, if G acts freely on Z, then $Z \to Z/G$ is a G-torsor.

Torsors in sheaves or alg. spaces

Instead of a scheme $E \to X$ we could consider a sheaf of sets on the site $(LFT/X)_{fl}$.

An action of G on a sheaf of sets E is $\forall\, T \to X$ an action of $G(T)$ on $E(T)$ compatible w/ restrictions. $T' \to T$.

A scheme E gives representable sheaf $h_E = Hom_X(-, E)$.

A sheaf E w/ G-action is a torsor if $(*)$ holds

(i.e. $E|_{U_i} \cong h_{G|_{U_i}}$ $\forall i$ $G|_{U_i}$-equivariantly)

$$PHS(G/X) := PHS_{sch}(G/X) \hookrightarrow PHS_{\substack{alg\\sp}}(G/X) \overset{\cong}{\hookrightarrow}_{Artin^{+}} PHS_{shv}(G/X) \cong \check{H}^1(X, G)$$

iso in many cases

Thm (III 4.3) A sheaf E on $X_{fl} = (LFT/X)_{fl}$ that is a G-torsor (i.e. an element of $PHS_{shv}(G/X)$) is repr by a scheme if

(a) G affine over X; or

(b) G smooth + sep over X, $\dim X \leq 1$; or

(c), –, (e)

pf. (a) flat descent of affine schemes
(b)–(e) more complicated descent arguments. [many results and counter-examples by Raynaud]

$+$ Artin: flat descent of any alg. space.

Non-abelian cohomology

$H^n(X, G)$ only makes sense for abelian sheaves G.

$\check{H}^1(X, G)$ makes sense for non-abelian G.

Recall that $H^1 = \check{H}^1$ in the abelian case.

For non-abelian G: $\mathcal{U} = (U_i \to X)_i$ covering

$$\check{H}^1(\mathcal{U}/X, G) = \text{coh. classes} = \text{"1-cocycles}/\text{1-coboundaries"}$$

$$\text{1-cocycle} = \left\{ (g_{ij}) : g_{ij} \in G(U_{ij}) \text{ satisfying cocycle condition } g_{ij} \cdot g_{jk} = g_{ik} \text{ after restriction to } U_{ijk} = U_i \times_X U_j \times_X U_k \right\}$$

$$\text{1-coboundary} = \{ (h_i) : h_i \in G(U_i) \}$$

$$(g_{ij}) \overset{(h_i)}{\sim} (g'_{ij}) \quad \text{if} \quad g'_{ij} = (h_i)|_{U_{ij}} \, g_{ij} \, (h_j)|_{U_{ij}}^{-1}$$

(cf. abelian case: $g'_{ij} = g_{ij} + h_i - h_j$)

$$\check{H}^1(X, G) = \varinjlim_{\mathcal{U}/X} \check{H}^1(\mathcal{U}/X, G)$$

Not groups, pointed sets: distinguished class $(1)_{ij}$.

SES $\quad 1 \to G' \to G \to G'' \to 1$ of groups $\Rightarrow$

LES $\quad 1 \to G'(X) \to G(X) \to G''(X) \xrightarrow{d} \check{H}^1(X, G') \to \check{H}^1(X, G) \to \check{H}^1(X, G'')$

of pointed sets.

Description of d: $g'' \in G''(X)$ locally lifts to $g_i \in G(U_i)$. Over intersections U_{ij}

$g_{ij} := g_i^{-1} g_j$ maps to zero in G'', hence gives $g_{ij} \in G'(U_{ij})$.

Classifying torsors

<u>Prop</u> (III 4.6) $PHS_{sh}(G/X) \xrightarrow{\cong} \check{H}^1(X, G)$

$$\begin{aligned} E \text{ sheaf torsor} &\longmapsto c(E) \\ \text{trivial torsor} &\longmapsto (1)_{ij} \end{aligned}$$

If E trivialized by $(U_i \to X)$, then we can pick $s_i \in E(U_i)$ $\forall i$.

Over U_{ij}, $\exists$ unique $g_{ij} \in G(U_{ij})$ s.th. $s_i g_{ij} = s_j$ (b/c $E|_{U_{ij}} \cong G|_{U_{ij}}$)

$c(E) = (g_{ij})$. Different choices of s_i give equivalent classes.

<u>Inverse map</u>:

$(g_{ij}) \in \check{H}^1(\mathcal{U}/X, G)$. Construct E by SES

$$1 \to E \to \prod \pi_{i*}(G|_{U_i}) \xrightarrow{d} \prod \pi_{ij*}(G|_{U_{ij}})$$
$$(h_i) \longmapsto g_{ij}^{-1} h_i^{-1} h_j$$

$E := d^{-1}(1)$.

d is G-equivariant under right action $(h_i, g) \mapsto (g^{-1} h_i)$

$\Rightarrow$ E has a G-action, i.e., a sheaf of sets w/ G-action.

If $(g_{ij}) = (1)$, then $E \cong G$.

More generally, if $(g_{ij}) = (g_i)(g_j)^{-1}$ for some (g_i), then also $E \cong G$
i.e. $(g_{ij}) \sim (1)$

$$\left(\begin{aligned} G &\longrightarrow \prod \pi_{i*}(G|_{U_i}) \\ h &\longmapsto h^{-1} g_i \end{aligned} \quad \text{gives } G \xrightarrow{\cong} E \right)$$

Locally, $(g_{ij}) \sim (1)$ so locally $E \cong G$, i.e., E is a G-torsor.

Remarks

(1) for G smooth, can define $\check{H}^1$ using Étale cohomology:

$$PHS_{shv}(G/X) \cong \check{H}^1(X_{Et}, G) \cong \check{H}^1(X_{fl}, G)$$

(2) for G abelian, addition in PHS given by contraction product

$$E_1 \overset{G}{\times} E_2 = E_1 \times E_2 \big/_{\varphi} G \quad \text{where } G\text{-action of quotient}$$

is $\varphi((e_1, e_2), g) = (e_1 g^{-1}, e_2 g)$ (requires G abelian)

and G-action on $E_1 \overset{G}{\times} E_2$ is

$$(e_1, e_2) g = (e_1 g, e_2) = (e_1, e_2 g)$$

Special groups

G special (in Serre's sense 1958) if every G-torsor is Zariski-locally trivial.

Prop: GL_n, $\mathbb{G}_m$, $\mathbb{G}_a$, SL_n, connected solvable, ... are special
Extensions of special groups are special.

Rmk: G special $\Rightarrow$ $\check{H}^1(X_{fl}, G) = \check{H}^1(X_{Et}, G) = \check{H}^1(X_{Zar}, G)$

In particular: $\operatorname{Pic} X = \check{H}^1(X_{Zar}, \mathbb{G}_m) = \check{H}^1(X_{fl}, \mathbb{G}_m) = H^1(X_{fl}, \mathbb{G}_m)$

Kummer theory

Kummer sequence:

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{\wedge n} \mathbb{G}_m \to 0$$

gives

$$0 \to \mu_n(X) \to \Gamma(X, \mathcal{O}_X^*) \xrightarrow{\wedge n} \Gamma(X, \mathcal{O}_X^*) \to$$

$$\to H^1(X, \mu_n) \to \operatorname{Pic} X \xrightarrow{\cdot n} \operatorname{Pic} X$$

$$H^1(X, \mu_n) = \{ (L, \phi) : L \in \operatorname{Pic} X, \ \phi: \mathcal{O}_X \xrightarrow{\cong} L^{\otimes n} \} / \sim$$

$(L, \phi) \sim (L', \phi')$ if $\exists$
$$\begin{tikzcd} & L^{\otimes n} \arrow[d, "\alpha^{\otimes n}"] \\ \mathcal{O}_X \arrow[ur, "\phi"] \arrow[dr, "\phi'"'] \arrow[r, phantom, "\circ"] & L'^{\otimes n} \end{tikzcd}$$
for $\alpha: L \xrightarrow{\sim} L'$

μ_n-torsor of (L, ϕ) is:

$$\operatorname{Spec}_X \underbrace{(\mathcal{O}_X \oplus L \oplus \cdots \oplus L^{\otimes n-1})}_{\substack{\mathbb{Z}/n\mathbb{Z}\text{-graded } \underline{\text{algebra}} \\ \Leftrightarrow \mu_n\text{-action}}}$$

mult: $L^{\otimes i} \otimes L^{\otimes j} \xrightarrow{can} L^{\otimes(i+j)}$ if $i+j<n$

$L^{\otimes i} \otimes L^{\otimes j} \xrightarrow{can} L^{\otimes(i+j)} \xrightarrow{\phi^{-1}} L^{\otimes(i+j-n)}$ if $i+j \geq n$

If L <u>trivial</u>: $L = \mathcal{O}_X \cdot e$, then

$$\mathcal{O}_X \oplus L \oplus \cdots \oplus L^{\otimes n-1} = \mathcal{O}_X[e]/(e^n - a) \qquad a = \phi^{-1}(e^{\otimes n}) \in \mathcal{O}_X^*$$

Etale-locally, a has an n^{th} root and then $\mathcal{O}_X[e]/(e^n-a) = \mathcal{O}_X[z]/z^n-1$
(Flat-locally in char p if $p|n$)

$z = e/a^{1/n}$

is isomorphic to μ_n. $\Rightarrow$ μ_n-torsor.

Ex: X complete variety $/k$ (i.e. geom irr + red). Then $\Gamma(X, \mathcal{O}_X) = k$.
and $\Gamma(X, \mathcal{O}_X^*) = k^*$

If in addition $k = \bar{k}$, then $k^* \xrightarrow{\wedge n} k^*$ surjective.

$$\Rightarrow H^1(X, \mu_n) = (\mathrm{Pic}\, X)_n := \ker(\mathrm{Pic}\, X \xrightarrow{n} \mathrm{Pic}\, X)$$

$\| \longleftarrow$ if char $k \nmid n$ since $k = \bar{k}$ has all nth roots of unity.

$H^1(X, \mathbb{Z}/n\mathbb{Z})$

If in addition X smooth curve of genus g, then

$$\mathrm{Pic}\, X = \mathrm{Pic}^0(X) \oplus \mathbb{Z} = \mathrm{Jac}(X)(k) \oplus \mathbb{Z}$$

When $p \nmid n$, then $\mathrm{Jac}(X)(k)_n \cong (\mathbb{Z}/n\mathbb{Z})^{2g}$ b/c $\mathrm{Jac}\, X$ abelian variety of dim g.

So $H^1(X, \mathbb{Z}/n\mathbb{Z}) = (\mathbb{Z}/n\mathbb{Z})^{2g}$ when X smooth curve of genus g and $p \nmid n$.

Agrees w/ calculation of prime-to-p fund. grp of X

$$\pi_1^{\acute{e}t}(X, \bar{x})^{p'} \approx \hat{\Gamma}_g^{p'} \qquad \Gamma_g = \frac{\text{free grp on } \alpha_1, \beta_1, \alpha_2, \beta_2, \ldots, \alpha_g, \beta_g}{(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}) \cdot \cdots \cdot (\alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1})}$$

$\nearrow$ prime-to-p part of profinite completion

(uses comparison in char 0 and specialization to char p)

Artin-Schreier theory

char $k = p$.

Artin-Schreier sequence

Artin-Schreier operator

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \to 0$$

$$x \longmapsto x^p - x$$

gives LES:

$$\to \Gamma(X, \mathcal{O}_X) \xrightarrow{F-1} \Gamma(X, \mathcal{O}_X) \to H^1(X, \mathbb{Z}/p\mathbb{Z}) \to H^1(X, \mathcal{O}_X) \xrightarrow{F-1} H^1(X, \mathcal{O}_X)$$

so we have SES:

$$0 \to \Gamma(X, \mathcal{O}_X)/_{(F-1)\Gamma(X,\mathcal{O}_X)} \to H^1(X, \mathbb{Z}/p\mathbb{Z}) \to H^1(X, \mathcal{O}_X)^F \to 0$$

$$\{x : Fx = x\}$$

If X affine, $H^1(X, \mathcal{O}_X) = 0$ and any $\mathbb{Z}/p\mathbb{Z}$-torsor is of the form

$$\mathrm{Spec}_X(\mathcal{O}_X[T]/T^p - T + a) \circlearrowleft \mathbb{Z}/p\mathbb{Z} \ni n \quad n.T = T + n$$

where $a \in \Gamma(X, \mathcal{O}_X)$. It is trivial $\iff a = b^p - b$ for some $b \in \Gamma(X, \mathcal{O}_X)$

(explicitly: coord change $T \longmapsto T + b = U$ gives ~~Sp~~$(\mathcal{O}_X[U]/U^p - U + 1) \cong \mathbb{Z}/p\mathbb{Z}$.)

For general X, $\exists$ open covering $(U_i \to X)$ and $a_i \in \Gamma(U_i, \mathcal{O}_X)$ and $f_{ij} \in \Gamma(U_{ij}, \mathcal{O}_X)$ s.th. $a_j - a_i = f_{ij}^p - f_{ij}$. Then (f_{ij}) determines a 1-cocycle and its class in $H^1(X, \mathcal{O}_X)$ satisfies $c(f_{ij})^p = c(f_{ij})$ since $c(a_j - a_i) = 0$. This describes:

$$H^1(X, \mathbb{Z}/p\mathbb{Z}) \longrightarrow H^1(X, \mathcal{O}_X)^F$$

$$(a_i), (f_{ij}) \longmapsto c(f_{ij})$$

Ex: X complete variety $/k \Rightarrow \Gamma(X, \mathcal{O}_X) = k$.

If k separably closed, then $k/(F-1)k = 0$. Thus

$$H^1(X, \mathbb{Z}/p\mathbb{Z}) \cong \underbrace{H^1(X, \mathcal{O}_X)^F}$$

k-v.sp of dim g if X curve of genus g.

k alg closed of char p.

Lemma: If V f.dim k-v.sp w/ action of Frobenius F (i.e., action by a semi-linear operator: $F(v+w) = F(v) + F(w)$ and $F(av) = a^p F(v)$)

Then $V \cong V_s \oplus V_n$ F-equivariantly where F acts bijectively on V_s and nilpotent on V_n. Moreover, V_s has basis $e_1, \dots, e_\sigma$ s.th. $F(e_i) = e_i$. Thus

$$V^F = V_s^F = \{\Sigma a_i e_i : a_i \in \mathbb{F}_p\}$$

and $F-1 : V \to V$ is surjective.

Applied to $F \circlearrowright H^1(X, \mathcal{O}_X)$ we obtain that $H^1(X, \mathcal{O}_X)^F$ f.dim $\mathbb{F}_p$-v.sp. of dimension σ where $\sigma \le \dim_k H^1(X, \mathcal{O}_X)$. That is

$$H^1(X, \mathbb{Z}/p\mathbb{Z}) \quad p\text{-group of order } p^\sigma, \text{ i.e., } \quad H^1(X, \mathbb{Z}/p\mathbb{Z}) = (\mathbb{Z}/p\mathbb{Z})^\sigma$$

Case X curve of genus $g \Rightarrow \sigma \le g$. ($\sigma$ = rank of Hasse-Witt matrix)

(cf. w/ $H^1(X, \mathbb{Z}/\ell\mathbb{Z}) = (\mathbb{Z}/\ell\mathbb{Z})^{2g}$ and $\sigma \le g < 2g$ if $g \ge 1$) if $\ell \ne p$.