# Étale cohomology #12
2016-04-27



## Comparison of topologies

Recall $(\mathbb{C}/X)_E$, $\mathbb{C}/X \subset Sch/X$ and $E$ topology

(or rather, $E$ class of morphisms s.t. $(U_i \xrightarrow{g_i} X)_i$ covering iff $(U_i \to X) \in E$ and $\bigcup g_i(U_i) = X$. Also, $\mathbb{C}/X$ closed under compositions w/ morphisms in $E$)

Ex: Small sites $X_{ét} = (ét/X)_{ét}$ $\quad$ Big sites $X_{Ét} = (LFT/X)_{ét}$

$X_{Zar} = (Zar/X)_{Zar}$ $\quad$ $X_{ZAR} = (LFT/X)_{Zar}$

$X_{fl} = (LFT/X)_{fl}$

## Comparison big vs small

III.3.1
**Prop**: $\mathbb{C}''/X \subset \mathbb{C}'/X$, $E$. Inclusion $\mathbb{C}''/X \subset \mathbb{C}'/X$ gives natural morphism of sites:

$$f : (\mathbb{C}'/X)_E \longrightarrow (\mathbb{C}''/X)_E$$

(a) $f_*$ is exact

(b) $f^* : Sh(\mathbb{C}''/X)_E \to Sh(\mathbb{C}'/X)_E$ fully faithful

Equivalently, $F \to f_* f^* F$ is an isom.

(c) $H^i(X, f_* F') \xrightarrow{\sim} H^i(X, F')$ $\quad \forall i, \ \forall F' \in Sh(\mathbb{C}'/X)_E$

(d) $H^i(X, F) \xrightarrow{\sim} H^i(X, f^* F)$ $\quad \forall i \ \forall F \in Sh(\mathbb{C}/X)_E$

Ex: $f : X_{Ét} \to X_{ét}$

$f^* : Sh(X_{ét}) \hookrightarrow Sh(X_{Ét})$ fully faithful

" $H^i(X_{Ét}) = H^i(X_{ét})$ "

Rmk: $G \to X$ ~~abelian~~ commutative group scheme

$\Rightarrow$ repr. sheaf $G = h_G \in Sh(X_{Ét})$

$f^* f_* G \to G$ iso.

$\Updownarrow$

$G \to X$ étale.

<u>pf</u>: (a) obvious since exactness at $U \longrightarrow X \in \mathbb{C}/X$ only depends on $(\mathbb{E}/U)_E$.

(b) For any $U \to X$ in $\mathbb{C}/X$ and $F \in Sh(\mathbb{C}/X)_E$:

$$\Gamma(U, f_* f^* F) \overset{\text{def}}{=} \Gamma(U, f^* F) \underset{\nearrow}{\cong} \Gamma(U, f^p F) \underset{(*)}{\cong} \Gamma(U, F)$$

sheafification over $U$ only depends on $(\mathbb{E}/U)_E$

$$(*) \quad \Gamma(U, f^p F) = \varinjlim_{\substack{U \xrightarrow{g} V \\ \searrow \ \swarrow \in \mathbb{C}/X \\ X}} \Gamma(V, F) = \Gamma(U, F) \quad \text{b/c} \quad \begin{array}{c} U \xrightarrow{id} U \\ \searrow \ \swarrow \\ X \end{array}$$

initial obj. in index category of limit.

(c)
$$\begin{tikzcd} Sh(\mathbb{C}'/X)_E \arrow[r, "f_*"', "\text{exact}"] \arrow[rr, bend right, "\Gamma"'] & Sh(\mathbb{C}/X)_E \arrow[r, "\Gamma"] & Ab \end{tikzcd}$$

and $f_*$ preserves injectives ($f^*$ exact)

(d)
$$\begin{tikzcd} H^i(X, F) \arrow[r] \arrow[rr, bend right, "\sim"'] & H^i(X, f^* F) \arrow[r, "\sim", "(c)"'] & H^i(X, f_* f^* F) \end{tikzcd}$$

(b): $F \xrightarrow{\sim} f_* f^* F$

## Changing E

**Rmk**: $u: \mathbb{C}_2/X \subset \mathbb{C}_1/X$, $E_2 \subset E_1$, $\Rightarrow$ <u>continuous morphism</u>

$$f: (\mathbb{C}_1/X)_{E_1} \longrightarrow (\mathbb{C}_2/X)_{E_2}$$

**Def**: $u$ is <u>cocontinuous</u> if for every $U \longrightarrow X$ in $\mathbb{C}_2$ and every covering $\{V_i \longrightarrow U\} \in E_1$ $\exists$ a covering $\{U_j \longrightarrow U\} \in E_2$ refining $\{V_i \longrightarrow U\}$. (i.e. $V_i \nearrow U_j \searrow X$, $V_i \longrightarrow X$ for every $i$ $\exists j$ s.t.)

**Prop III 3.3**: $u$ cocontinuous $\Rightarrow$ $f_*$ exact. Hence $H^i(X_{E_2}, f_*F) \xrightarrow{\sim} H^i(X_{E_1}, F)$

**pf**: Let $(U \longrightarrow X) \in \mathbb{C}_2/X$, and $F \longrightarrow F'' \longrightarrow 0$ exact on $Sh(\mathbb{C}_1/X)_{E_1}$.

Thus $\forall s'' \in F''(U)$, $\exists$ cov. $\{V_i \to U\} \in E_1$ and lifts $s_i \in F(V_i)$.

After refining $\{V_i \to U\}$ to $\{U_j \to U\} \in E_2$ we obtain lifts $s_j \in F(U_j)$

so $F \longrightarrow F'' \longrightarrow 0$ also exact on $Sh(\mathbb{C}_2/X)_{E_2}$.

**Rmk**: More abstract explanation: cocontinuity gives a morphism of topoi in the other direction:

$$(f_*, f^!): Sh(\mathbb{C}_2/X)_{E_2} \longrightarrow Sh(\mathbb{C}_1/X)_{E_1}$$

so have adjoints $\underset{u_!}{\overset{\shortparallel}{f^*}}, \underset{u^*}{\overset{\shortparallel}{f_*}}, \underset{u_*}{\overset{\shortparallel}{f^!}}$ $\Rightarrow$ $f_*$ exact

**Prop** (Stacks Proj 03A0): If $u$ cocontinuous and $\forall (V \longrightarrow X) \in \mathbb{C}_1/X$ $\exists$ covering $\{V_j \longrightarrow V\} \in E_1$ w/ $V_j \in \mathbb{C}_2/X$, then $f_*$ <u>equiv. of cat.</u>

<u>Examples</u>: (étale) & smooth

$E_0$ = all <u>smooth</u>
$\cup$
$E_1$ = all étale
$\cup$
$E_2$ = étale + finite type = Milne's (Ét)
$\cup$
$E_3$ = separated + étale + finite type ($\Leftrightarrow$ q-aff + étale) ZMT
$\cup$
$E_4$ = affine + étale

Also $(E_0/X)_{E_0} \xrightarrow{g_0} (E_1/X)_{E_1} \xrightarrow{g_1} \cdots$ and $(g_0)_*, \dots, (g_3)_*$ equiv of cat.
$(g_0)_*$ <u>not</u> equiv of cat, but $(g_0)_*$ exact.

Morphisms of sites

$$(LFT/X)_{E_0} \xrightarrow{f_0} (LFT/X)_{E_1} \xrightarrow{f_1} \cdots \xrightarrow{f_3} (LFT/X)_{E_4}$$

$(f_0)_*$, $(f_1)_*$, $(f_2)_*$ equiv of categories

$(f_3)_*$ equiv. if $X$ separated: refine étale $\{V_j \to U\}$ by $\{\mathrm{Spec}(A_i) \to U\}$ where $V_j = \bigcup \mathrm{Spec}\, A_i$ open cov for some $i$.

Proves $(f_1)_*, (f_2)_*, (f_3)_*$ equiv. For $(f_0)_*$ use:

<u>Thm</u> (quasi-sections of smooth): I.3.26 $V \to U$ smooth surj. $\exists\, U' \to U$ étale surj and

$$\begin{tikzcd} & V \arrow[d] \\ U' \arrow[ur] \arrow[r] & U \end{tikzcd}$$

<u>More examples</u>: (flat)

$E_1$ = all flat + loc. fin. type (better: loc. of fin. pres.) non-noeth case
$\cup$
$E_2$ = flat + fin. type = Milne's (fl)
$\cup$
$E_3$ = quasi-finite + flat
$\cup$
$E_4$ = affine + flat + fin. type $\quad$ $E_5$ = finite + flat

$E_1 - E_3$ gives equiv of cat. (for $E_2 \subset E_3$ use quasi-finite sections of flat I.2.25). Also $E_1 \subseteq E_4$ if $X$ separated.

$E_3 \subset E_5$ gives equiv if $X$ henselian (use qf + sep = finite $\amalg$ q-fin)

# Flat vs Zariski

$\mathcal{F} \in QCoh(X)$.

$W(F) \in Sh(X_{fl})$ $\qquad \Gamma(U \xrightarrow{g} X, W(F)) = \Gamma(U, g^{-1}F \otimes_{\mathcal{O}_X} \mathcal{O}_U)$

Lemma (II.1.6) $W(F)$ is a sheaf.

pf: Since sheaf in big Zariski site, enough to verify sheaf condition for coverings $U' \to U$ w/ $U = \mathrm{Spec}\, A$, $U' = \mathrm{Spec}\, B$. This is the exactness of

$$0 \to M \to M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

for $A \to B$ faithfully flat and $M$ any $A$-module.

(sketch pf: tensor w/ $\otimes_A B$, then use section)

Prop (III 3.7) $f : X_{fl} \to X_{Zar}$, $F \in QCoh(X)$. There are canonical iso:

$$H^i(X_{Zar}, F) \to H^i(X_{fl}, W(F)) \quad \forall i$$

pf: Leray spectral sequence:

$$H^i(X_{Zar}, R^j f_* W(F)) \Rightarrow H^{i+j}(X_{fl}, W(F))$$

Since $f_* W(F) = F$, enough to prove that $R^j f_* W(F) = 0 \quad \forall j > 0$.

$R^j f_* W(F)$ is the sheaf ass. to presheaf

$$U \longmapsto H^j(U_{fl}, W(F)) \qquad \text{on } X_{Zar}$$

Thus enough to prove that:

$$H^j(U_{fl}, W(F)) = 0 \quad \forall j > 0 \text{ when } \underline{U \text{ affine}}.$$

By comparison of topologies (changing $E$) we may work in the topology $E$ = affine flat morphisms. And small site $\mathcal{U}_{E(f)}$ ← finite $E$-covers $(U_i' \to U)_{i=1}^n$ b/c $U$ q-cpt.

Recall $F|_U$ flabby if TFE holds

(i) $H^i(U', F) = 0 \; \forall i > 0$, $U' \to U$ in $C/U$

(ii) $\check{H}^i(\mathcal{U}'/U', F) = 0 \; \forall i > 0$, $U' \to U$ in $C/U$ and covering $\mathcal{U}'/U'$.

Thus enough to prove that $F|_U$ flabby. Enough to verify

$$\check{H}^i(\mathcal{U}'/U', F) = 0 \quad \forall i > 0, \quad U' = \operatorname{Spec} A \xrightarrow{\text{flat}} U$$

$$\mathcal{U}' = (U_i'' = \operatorname{Spec} B_i \to \operatorname{Spec} A)_{i=1}^n$$

We may replace $\mathcal{U}'$ with $\left(\coprod_{i=1}^n U_i'' \to \operatorname{Spec} A\right) = (\operatorname{Spec} \underbrace{\prod_{i=1}^n B_i}_{B :=} \to \operatorname{Spec} A)$

Then Čech coh is cohomology of complex

$$M \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \longrightarrow M \otimes_A B \otimes_A B \otimes_A B \longrightarrow \cdots$$

$(F|_U = \tilde{M})$ which is exact (except at $i=0$ where $H^0 = M$).

similar argument as in Lemma basic of flat descent result.

□

Rmk: Same pf shows:

$$H^i(X_{Zar}, F) = H^i(X_{Et}, \omega(F))$$

# Flat vs étale cohomology

**Thm III 3.9** $G$ smooth commutative group scheme (or alg space) over $X$. Then canonical isomorphism:

$$H^i(X_{\acute{e}t}, G) \xrightarrow{\sim} H^i(X_{fl}, G)$$

pf: $f : X_{fl} \to X_{\acute{e}t}$. Leray s.s.

$$H^i(X_{\acute{e}t}, R^j f_* G) \Rightarrow H^{i+j}(X_{fl}, G)$$

As before, enough to prove $R^j f_* G = 0$ $\forall j > 0$.

Enough to prove $(R^j f_* G)_{\bar{x}} = 0$ $\forall$ geom pts $\bar{x}$.

We have proved that

$$(R^j f_* G)_{\bar{x}} = H^j((X_{\bar{x}})_{fl}, G)$$

We can thus assume that $X$ strictly local and want to prove that $H^j(X, G) = 0$ $\forall j > 0$.

Lemma: $H^i(X_E, F) = H^i(X_{fl}, F)$ for all $F$ on $X_{fl}$ when $X$ strictly local and $E$ = finite flat morphisms.

pf: Have seen that $X_{fl}$ can be replaced w/ $X_{qff}$.

A quasi-finite separated morphism $U \to X$ splits as $U'_* \amalg U''_* \to X$

where $U'_* \to X$ finite and $U''_* \to X$ misses closed point.

If $(U_i \to X)_i$ qff covering, then $U'_i \to X$ finite, flat, surj for some $i$. $\square$

~~Lemma~~

It's enough to prove that $G$ flabby w.r.t. E-topology, i.e.

$$\check{H}^i(X'/X, G) = 0 \quad \forall X \text{ strictly local and } X' \to X \text{ finite, faithfully flat.}$$

Step 1: $\check{H}^i(X'/X, G) = \check{H}^i(X'_0/X_0, G)$ where $X_0 \hookrightarrow X$ closed pt and $X'_0 = X' \times_X X_0$.

Step 2: $\check{H}^i(X'_0/X_0, G) = 0 \quad \forall i > 0$.

Step 1 using that for smooth maps $Z \to X$, sections of $Z_0 \to X_0$ lifts to sect. of $Z \to X$.

But several technical constructions.

Ex: $G = \mathbb{G}_a, \mathbb{G}_m, \mu_r, \underline{\mathbb{Z}/a\mathbb{Z}}$ (étale = flat coh.)

non-ex: $G = \alpha_p, \mu_p$ (non-smooth grps) (étale $\neq$ flat coh.)

More smooth ex: $G = \underline{A}$ constant sheaf $A$ abelian.

$G$ locally constant ($\Leftrightarrow G \to X$ finite étale if $G_{\bar{x}}$ finite $\forall \bar{x}$)

$G = f^*F$, $F \in Sh(X_{\acute{e}t})$ "$G$ locally constructible"

$\Leftrightarrow G \to X$ $\underbrace{\text{étale algebraic space}}_{\text{commutative group}}$

# Étale vs complex cohomology

**Thm** (III 3.12) $X$ smooth$/\mathbb{C}$, $A$ finite abelian. Then

$$H^i(X^{an}, A) \cong H^i(X_{et}, A)$$

**Rmk**: More generally (SGA4, Exp XVI, Thm 4.1) holds for:

* $X$ singular, finite type$/\mathbb{C}$
* ~~[illegible]~~
* $A$ constructible sheaf of torsion abelian grps. (e.g. $A$ torsion and locally constant)
* for $R^if_*$ instead of $H^i$.

**Outline of pf**:

① Introduce site $X_{cx}$ (also called $X_{cl}$): objects $U \longrightarrow X^{an}$ local iso.
covers: jointly surjective.

Then morphism $X_{cx} \longrightarrow X^{an}$ induces equiv. on sheaves so $H^i(X_{cx}) = H^i(X^{an})$

$\uparrow$ classical site of top. space $X^{an}$ obj: $U \longrightarrow X^{an}$ open subset.

Obtains morphism of sites $f: X_{cx} \longrightarrow X_{et}$.

Leray S.S.

$$H^i(X_{et}, R^jf_*F) \Rightarrow H^{i+j}(X_{cx}, F) = H^{i+j}(X^{an}, F)$$

Enough to prove that $R^jf_*F = 0$ $\forall j > 0$, $F$ loc. const. torsion.

(*) Enough to show that given $\gamma \in H^i(X_{cx}, F)$ $\exists U \longrightarrow X$ étale cov. s.th $\gamma|_{U_{cx}} = 0$.

## ② Riemann existence / Grauert–Remmert. III.3.14

$$\mathrm{F\acute{E}t}(X) \longrightarrow \mathrm{F\acute{E}t}(X_{\mathbb{C}})$$

$$\underset{\text{finite étale}}{U \to X} \longmapsto \underset{\text{finite étale}}{U_{\mathbb{C}}^{an} \to X^{an}}$$

equivalence of categories. In particular, theorem and (*) holds for $i=1$. (b/c $H^1$ classifies torsors under $A$ and these are finite étale)

pf: (SGA4 XI, Thm 4.3) reduce to $X$ affine and use Grauert–Remmert and GAGA ~~for~~ $\mathrm{Coh}(\bar{X}) \xrightarrow{\sim} \mathrm{Coh}(\bar{X}^{an})$ for $\bar{X}$ proj.

or (SGA1 XII, Thm 5.1) use res. of sings and GAGA for Coh.

## ③ Artin's good nbhds

**Thm**: (SGA4, XI, Prop 3.3) $X/k$ smooth, $k=\bar{k}$. Then $\forall x \in X$ $\exists U \subset X$ nbhd s.th. $U$ is a good nbhd, i.e., $\exists$

$$U = U_n \longrightarrow U_{n-1} \longrightarrow \cdots \longrightarrow U_1 \longrightarrow \mathrm{Spec}\, k$$

where $U_k \longrightarrow U_{k-1}$ an elementary fibration.

**Def.** $f: X \longrightarrow S$ elementary fibration if $\exists$

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[dr, "f"'] & \bar{X} \arrow[d, "\bar{f}"] & Y \arrow[l, hook', "i"'] \arrow[dl, "g"] \\ & S & \end{tikzcd}$$

s.th.

(i) $j$ open imm, $X = \bar{X} \setminus Y$, $i$ closed imm.

(ii) $\bar{f}$ proj smooth, family of conn. curves.

(iii) $g$ surjective, finite étale.

④ Proper base change in ordinary topology (easy result)

Thm: $X \xrightarrow{\pi} S$ proper map of locally compact top spaces.

Then $(R^i\pi_* \mathcal{F})_s \simeq H^i(X_s, \mathcal{F})$ for any sheaf $\mathcal{F}$ on $X$.

See Milne LEC 17.2 or Iversen "Cohomology of sheaves" Thm III 6.2

⑤ Proper-smooth base change in topology (easy)

Thm: $X \xrightarrow{\pi} S$ proper, smooth (i.e. submersion) map of manifolds. Then $R^i\pi_*\mathcal{F}$ locally constant for any locally constant $\mathcal{F}$ on $X$ ("local systems")

pf: By Ehresmann's fibration theorem, $\pi$ is a fibration. Thus: WLOG $S$ contractible, $X = Z \times S$, $\mathcal{F} = \pi_1^* \mathcal{F}_Z$. Then $R^i\pi_*\mathcal{F}$ = constant sheaf w/ stalk $H^i(Z, \mathcal{F}_Z)$. □

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Apply Leray's s.s. to Artin's good nbhds and use ④-⑤ to reduce to ② to prove (*).

Alt. proof (see SGA4, Exp XI, 4.6, due to Serre)

Prove that good nbhds are $K(\pi_1, 1)$:s.

Then prove that $\pi_1(X(\mathbb{C}))$ is a good group. (cf Lecture #1)