Étale cohomology #11

20/4 – 2016

Čech cohomology

X scheme, E Grothendieck topology on C/X, site X_E.

$\mathcal{U} = (U_i \xrightarrow{\phi_i} X)_{i \in I}$ covering of X (or replace X w/ any obj $U \to X$ in C/X)

Put $U_{i_0,\dots,i_p} := U_{i_0} \times_X \cdots \times_X U_{i_p}$ $\quad i_j \in I$

(in topology $U_i \subset X$ open and enough to consider $U_{i_0 < \dots < i_p} = U_{i_0} \cap \dots \cap U_{i_p}$)

P presheaf on X_E. Restriction maps:

$$U_{i_0,\dots,i_p} \xrightarrow[\mathrm{pr}_j]{} U_{i_0,\dots,\check{i}_j,\dots,i_p}$$

$$P(U_{i_0,\dots,\check{i}_j,\dots,i_p}) \xrightarrow[\mathrm{res}_j]{} P(U_{i_0,\dots,i_p})$$

Čech complex:

$$C^\cdot(\mathcal{U}, P) := (C^p(\mathcal{U}, P), d^p) \qquad \underset{s}{C^p(\mathcal{U},P)} \ni = \prod_{(i_0,\dots,i_p) \in I^{p+1}} P(U_{i_0,\dots,i_p})$$

$$(d^p(s))_{i_0,\dots,i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j \mathrm{res}_j (s_{i_0,\dots,\check{i}_j,\dots,i_p})$$

$$d^{p+1} d^p = 0$$

$\check{H}^p(\mathcal{U}, P)$ = coh. grp of $C^\cdot(\mathcal{U}, P)$ = Čech coh groups of P w.r.t. $\mathcal{U}$.

Rmk: $\check{H}^0(\mathcal{U}, P) = \ker\left(\prod_{i \in I} P(U_i) \rightrightarrows \prod_{i,j \in I} P(U_{ij})\right)$

Thus have natural map $P(X) \longrightarrow \check{H}^0(\mathcal{U}, P)$ which is iso if P is a sheaf.

<u>Refinement</u>: $\mathcal{V} = (V_j \xrightarrow{\psi_j} X)_{j\in J}$ another cover of X. $\mathcal{V}$ <u>refinement</u> of $\mathcal{U}$ if

$\exists \tau : J \longrightarrow I$ and commutative diagrams $(i = \tau(j))$

$$\begin{tikzcd} V_j \arrow[r, "\psi_j"] \arrow[d, "\eta_j"'] & X \\ U_i \arrow[ur, "\phi_i"'] & \end{tikzcd}$$

This gives $\tau^p : C^p(\mathcal{U}, P) \longrightarrow C^p(\mathcal{V}, P)$

$$s \longmapsto \left(\mathrm{res}_{\eta_{j_0}^{*} \times \cdots \times \eta_{j_p}} (s_{\tau(j_0), \dots, \tau(j_p)})\right)_{j_0, \dots, j_p}$$

Respects d so obtain:

$$\rho(\mathcal{V}, \mathcal{U}, \tau) : \check{H}^p(\mathcal{U}, P) \longrightarrow \check{H}^p(\mathcal{V}, P)$$

<u>Lemma</u>: $\rho(\mathcal{V}, \mathcal{U}, \tau)$ independent of τ and η_j.

If $\mathcal{W}$ refinement of $\mathcal{V}$, then $\rho(\mathcal{W}, \mathcal{U}) = \rho(\mathcal{W}, \mathcal{V}) \circ \rho(\mathcal{V}, \mathcal{U})$.

<u>Čech coh. grps</u>: $\check{H}^p(X_E, P) := \varinjlim \check{H}^p(\mathcal{U}, P)$

<u>similarly</u>: $\check{H}^p(U, P) := \varinjlim \check{H}^p(\mathcal{U}/U, P)$

Presheaf $\underline{H}^p(X, P)$: $U \longmapsto \check{H}^p(U, P)$

<u>Rmk</u>: In general $\check{H}^p(X, P)$ cannot be computed using $i_0 < \dots < i_p$ as in topology unless all $U_i \to X$ monomorphisms (e.g. open immersions).

<u>Ex</u>: $\mathcal{U} = (U \to X)$, $X = \operatorname{Spec} A$, $U = \operatorname{Spec} B$; $P = \mathbb{G}_m$. Then $C^\bullet(\mathcal{U}, P)$ is:

$$0 \longrightarrow B^* \longrightarrow (B \otimes_A B)^* \longrightarrow (B \otimes_A B \otimes_A B)^* \longrightarrow \cdots$$

S.E.S of presheaves $0 \to P' \to P \to P'' \to 0$ gives

SES of abelian grps: $0 \to C^\bullet(\mathcal{U}, P') \to C^\bullet(\mathcal{U}, P) \to C^\bullet(\mathcal{U}, P'') \to 0$ gives

LES of Čech cohomology groups:

$$0 \to \check{H}^0(P') \to \check{H}^0(P) \to \check{H}^0(P'') \to \check{H}^1(P') \to \check{H}^1(P) \to \check{H}^1(P'') \to \cdots$$

<u>Prop</u>: $\check{H}^p(\mathcal{U}, -) = R^p\check{H}^0(\mathcal{U}, -)$ where $\check{H}^0(\mathcal{U}, -) : PSh(X_E) \longrightarrow Ab$.

<u>pf</u>: Remains to verify that $\check{H}^p(\mathcal{U}, I) = 0$ for all $p > 0$ and I injective presheaf.

See Milne lemma III 2.4.

<u>Rmk</u>: Even when S is a sheaf, so that $\check{H}^0(X, S) = \underset{X(S)}{H^0(X, S)}$, $R^p\check{H}^0(X, S)$ need not equal $R^p H^0(X, S)$

<u>Reason</u>: H^p derived from $Sh(X_E) \longrightarrow Ab$.

<u>Goes wrong</u>: SES of <u>sheaves</u> need not give LES of Čech coh. grps. (only obstruction).

<u>Prop</u>: There exists a spectral seq.

$$E_2^{p,q} = \check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Longrightarrow H^{p+q}(X, \mathcal{F})$$

and $$\check{H}^p(X, \underline{H}^q(\mathcal{F})) \Longrightarrow H^{p+q}(X, \mathcal{F})$$

<u>pf</u>: F: Sheaves $\xrightarrow{\text{forget}}$ Presheaves, $G = \check{H}^0(\mathcal{U}, -)$: Presheaves $\longrightarrow Ab$

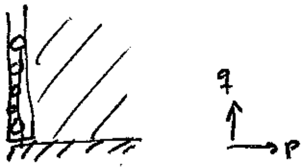
$F = \underline{H}^0 = i$

$$\begin{array}{c} GF : \text{Sheaves} \longrightarrow Ab \\ \parallel \\ \Gamma(X, -) \end{array}$$

Since $i(\text{inj.}) = \text{inj}$ we get a Grothendieck spectral seq. □

(b/c left adjoint a is exact)

<u>Prop</u>: $\check{H}^0(\mathcal{U}, \underline{H}^q(\mathcal{F})) = 0 \quad \forall q > 0$



<u>pf</u>: Take inj. res $F \longrightarrow I^\bullet$.

$$\underline{H}^q(F) = \text{coh. of } i(I^\bullet)$$

$$\check{H}^0(\mathcal{U}, \underline{H}^q(\mathcal{F})) = (a\,\underline{H}^q(\mathcal{F}))(\mathcal{U}) = a\,H^q(iI^\bullet) \overset{a \text{ exact}}{=} H^q(a i I^\bullet) = H^q(I^\bullet) = 0 \quad \square$$

<u>Cor</u>: $\left(\check{H}^0(\mathcal{U},\mathcal{F}) \cong H^0(\mathcal{U},\mathcal{F})\right)$

$$\check{H}^1(\mathcal{U},\mathcal{F}) \cong H^1(\mathcal{U},\mathcal{F})$$

<u>Rmk</u>: If $\mathcal{U}/\mathcal{U}$ covering s.th. $H^q(\mathcal{U}_{i_0,\dots,i_p},\mathcal{F}) = 0 \quad \forall i_0,\dots,i_p,\ q>0$

then $\check{H}^p(\mathcal{U}, \underline{H}^q(\mathcal{F})) = 0 \quad \forall q>0$ so spectral seq. gives

$$\check{H}^p(\mathcal{U},\mathcal{F}) \cong H^p(\mathcal{U},\mathcal{F})$$

(<u>Compare</u> (topology): Take covering by contractibles s.th. intersections are contractible)

<u>Prop</u>: $\mathcal{F}$ quasi-coherent sheaf of $\mathcal{O}_X$-modules

X_{Zar} Zariski topology, X <u>separated</u> then $\check{H}^p(X_{Zar},\mathcal{F}) \cong H^p(X_{Zar},\mathcal{F}) \quad \forall p$

<u>pf</u>: Take open covering by affines, then intersections are also affine (b/c X sep.)

Then $H^q(\mathcal{U}_{i_0,\dots,i_p},\mathcal{F}) = 0 \quad \forall q>0$ by following

<u>Lemma</u>: X affine, $\mathcal{F}$ q-coh $\mathcal{O}_X$-mod. $H^p(X_{Zar},\mathcal{F}) = 0 \quad \forall p>0$.

To prove that $\check{H}^p(X,-) = H^p(X,-)$ we need that a SES of sheaves gives a LES of Čech cohomology. The problem is that for a surjection $F \twoheadrightarrow F''$, then

$$F(U_{i_0,\dots,i_p}) \longrightarrow F''(U_{i_0,\dots,i_p})$$

is not nece surj, only locally surjective, ie. for any $s \in F''(U_{i_0,\dots,i_p})$ $\exists$ refinement $(V_j \to U_{i_0,\dots,i_p})$ s.th $s|_{V_j}$ lifts to F. If we can find a refinement of the form $V_{i_0,\dots,i_p} \longrightarrow U_{i_0,\dots,i_p}$ then

$$\varinjlim_{\mathcal{U}} \prod F(U_{i_0,\dots,i_p}) \longrightarrow \varinjlim_{\mathcal{U}} \prod F''(U_{i_0,\dots,i_p})$$

is surjective and $\check{H}^p = H^p$. This happens in the following case:

2.17 **Thm** (Artin) X quasi-compact scheme, s.th every finite subset contained in an open affine (e.g. X quasi-proj over affine), F sheaf on X_{Et}. Then $\check{H}^p(X_{Et}, F) \xrightarrow{\cong} H^p(X_{Et}, F)$.

After some limits, $\exists$ of refinements as above boils down to:

2.18 **Lemma:** A ring, $p_1,\dots,p_r$ prime ideals, $A' = A^{sh}_{p_1} \otimes_A A^{sh}_{p_2} \otimes_A \dots \otimes_A A^{sh}_{p_r}$, $A' \xrightarrow{\varphi} B$ faithfully flat, étale. Then φ has a section.

X integral qc scheme, $g : \operatorname{Spec} K \hookrightarrow X$ generic point

$$\phi : \mathbb{G}_{m,X} \hookrightarrow g_* \mathbb{G}_{m,K}$$

$$\Gamma(U, \mathcal{O}_U^*) \hookrightarrow R(U)^* \quad \text{rational functions on } U$$

$$(*) \quad 0 \to \mathbb{G}_{m,X} \to g_* \mathbb{G}_{m,K} \to \mathrm{Div}_X \to 0$$

sheaf of Cartier divisors

If X regular then $\mathrm{Div}_X \cong D_X = \bigoplus_{x \in X_1} i_{x*} \mathbb{Z}$ $\qquad X_1 =$ codim 1 points of X

sheaf of Weil divisors

Long-exact seq from $(*)$

$$\to H^r(X_{\acute{e}t}, D_X) \to H^{r+1}(X_{\acute{e}t}, \mathbb{G}_{m,X}) \to H^{r+1}(X_{\acute{e}t}, g_* \mathbb{G}_{m,K}) \to \cdots$$

Leray s.s. from $i_x : x \hookrightarrow X$

$$H^p(X, R^q i_{x*} \mathbb{Z}) \Rightarrow H^{p+q}(x, \mathbb{Z})$$

Galois cohomology computation: $H^0(x, \mathbb{Z}) = \mathbb{Z}$, $H^1(x, \mathbb{Z}) = 0$, $R^1 i_{x*} \mathbb{Z} = 0 \overset{\text{S.S.}}{\Longrightarrow}$

$$H^0(X, D_X) = \bigoplus_{x \in X_1} \mathbb{Z} \qquad H^1(X, D_X) = 0 \qquad \left(H^2(X, D_X) \neq 0 \text{ in general} \right)$$

Leray s.s for $g : \operatorname{Spec} K \hookrightarrow X$.

$$H^p(X, R^q g_* \mathbb{G}_m) \Rightarrow H^{p+q}(\operatorname{Spec} K, \mathbb{G}_m)$$

Hilbert's thm 90: $R^1 g_* \mathbb{G}_{m,K} = 0 \overset{\text{s.s.}}{\Rightarrow}$

$$H^0(X, g_* \mathbb{G}_{m,K}) = K$$

$$H^1(X, g_* \mathbb{G}_{m,K}) = 0$$

$$0 \longrightarrow \underset{\substack{\| \\ H^0(X, \mathbb{G}_m)}}{\Gamma(X, \mathcal{O}_X)^*} \longrightarrow K^* \longrightarrow \bigoplus_{x \in X_1} \mathbb{Z} \longrightarrow H^1(X, \mathbb{G}_m) \to 0$$

$$H^1(X_{et}, \mathbb{G}_m) = (\text{divisors}/\text{principal div.}) = \operatorname{Pic} X = H^1(X_{Zar}, \mathcal{O}_X^*)$$

<u>Ex</u>: X sm. curve, $k = \bar{k}$. Then $H^r(X, \mathbb{G}_m) = 0 \quad \forall r > 1$.