Étale cohomology #10

13/4 - 2016

Calculating $R^i\pi_*$

$\pi: X' \to X$, $\mathcal{F} \in Sh(X'_{ét})$.

Case 1: $X = \operatorname{Spec} \bar{k}$, $\bar{k}$ sep. closed.

$$\begin{tikzcd} Sh(X') \arrow[r, "\pi_*"] \arrow[rr, bend right, "\Gamma(X',-)"'] & Sh(X) \arrow[r, "\Gamma(X,-)", "\sim"'] & Ab \end{tikzcd}$$

so $R^i\pi_* \xrightarrow{\sim} \Gamma(R^i\pi_*) = H^i(X', -)$

Case 2: $X = \operatorname{Spec}(\mathcal{O}^h_{X,\bar{x}})$ strictly henselian. Then $\Gamma(X,-)$ exact so

$$(R^i\pi_*\mathcal{F})_{\bar{x}} \cong \Gamma(R^i\pi_*\mathcal{F}) = R^i(\Gamma\pi_*\mathcal{F}) = H^i(X,\mathcal{F})$$

General: $R^i\pi_*\mathcal{F}$ sheaf associated to presheaf $U \mapsto H^i(U', \mathcal{F}|_{U'})$

where $U' = U \times_X X'$. To be precise: $U \xrightarrow{ét} X \rightsquigarrow U' \xrightarrow{ét} X'$.

(presheaf is a sheaf for $i=0$).

pf (Milne III.1.13) $\pi_*: Sh(X'_{ét}) \xrightarrow{i} PSh(X'_{ét}) \xrightarrow{\pi_p} PSh(X_{ét}) \xrightarrow{a} Sh(X_{ét})$.

a and π_p are exact. Thus, if $\mathcal{F} \to I^\bullet$ injective resolution then

$$R^i\pi_*(\mathcal{F}) = H^i(a\pi_p i(I^\bullet)) = a\pi_p H^i(i(I^\bullet)) = a\pi_p \underline{H}^i(\mathcal{F}). \qquad \square$$

Stalks: Let $\bar{x} \to X$ geom. point.

Then $(R^i\pi_*\mathcal{F})_{\bar{x}} \xrightarrow{\sim} H^i(\tilde{X}', \tilde{\mathcal{F}})$

$$\begin{tikzcd} X' \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & \tilde{X}' = X' \times_X \tilde{X} \arrow[l] \arrow[d] \\ X & \tilde{X} = \operatorname{Spec} \mathcal{O}_{X,\bar{x}} \arrow[l] \end{tikzcd}$$

pf: $(R^i\pi_*\mathcal{F})_{\bar{x}} = (R^i\pi_p\mathcal{F})_{\bar{x}} = \varinjlim_{\substack{U' \xrightarrow{ét} U \\ \bar{x}}} H^i(U', \mathcal{F}|_{U'})$ $\qquad \tilde{X} = \varprojlim U'$

Lemma: $\tilde{X}' = \varprojlim_\alpha X'_\alpha \Rightarrow \varinjlim_\alpha H^i(X'_\alpha, \mathcal{F}|_{X'_\alpha}) = H^i(\varprojlim X'_\alpha, \mathcal{F}|_{\varprojlim X'_\alpha})$

<u>Rmk:</u> Compare w/ quasi-coherent cohomology where

$$R^i\pi_* \mathcal{F} = H^i(X', \mathcal{F})^\sim \quad \text{if } X \text{ affine}$$

No étale analogue except for $X = \operatorname{Spec} \bar{k}$.

<u>Rmk:</u> If π finite, then π_* <u>exact</u> and

$$(\pi_* \mathcal{F})_{\bar{x}} = \Gamma(X'_{\bar{x}}, \mathcal{F}|_{X'_{\bar{x}}}) = \prod_{\substack{X' \to X \\ \nwarrow \;\; \nearrow \\ \bar{x}}} \mathcal{F}_{\bar{x}} \qquad X'_{\bar{x}} = X' \times_X \operatorname{Spec} k(\bar{x}) \text{ fiber}$$

<u>Proper base change</u> (diff. Thm)

π <u>proper</u>, then

$$(R^i\pi_* \mathcal{F})_{\bar{x}} = H^i(X'_{\bar{x}}, \mathcal{F}|_{X'_{\bar{x}}}) \quad \forall \bar{x} \to X$$

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\pi"] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

In particular:

$$f^* R^i\pi_* \mathcal{F} \xrightarrow{\sim} R^i\pi'_* f'^* \mathcal{F}$$

$\forall$: $f: Y \longrightarrow X$

Spectral sequences

$$E_2^{p,q} \Rightarrow E_\infty^{p+q}$$

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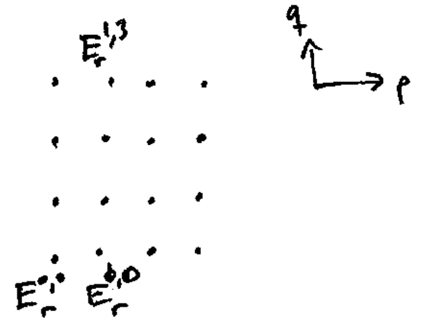
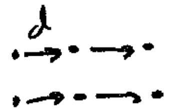


Differentials: page 1



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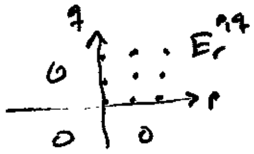
$$d_r^{p,q}: E_r^{p,q} \longrightarrow E_r^{p+r,\, q-r+1}$$

First quadrant



$$(d_r)^2 = 0$$

$$E_{r+1}^{p,q} = \text{cohomology of } E_r^{\bullet,\bullet} \text{ at } (p,q) = \ker(d_r^{p,q}) / \operatorname{im}(d_r^{p-r,\,q+r-1})$$

A S.S. is $(E_r^{\bullet,\bullet}, d_r)_{r \geq r_0}$ for some r_0. We write $E_{r_0}^{p,q} \Rightarrow E_\infty^{p+q}$.

$E_\infty^{p,q} = E_r^{p,q}$ stabilizes for $r \gg 0$ (differentials starting and ending at p,q outside 1^{st} quadrant)

$E_\infty^n = \bigoplus_{n=p+q} E_\infty^{p,q}$ if k-vector spaces. In general; an extension of the $E_\infty^{p,q}$'s:

E_∞^n has a filtration $0 = F^{n+1}E_\infty^n \subset F^n E_\infty^n \subset \cdots \subset F^0 E_\infty^n = E_\infty^n$

s.th. $E_\infty^{p,q} = \operatorname{gr}_p E_\infty^n = F^p E_\infty^n / F^{p+1} E_\infty^n$.

Consequences (examples)

- $E_r^{p,q}$ torsion $\Rightarrow E_{r+1}^{p,q}$ torsion $\Rightarrow E_\infty^n$ torsion

  fin. gen. — fin. gen — fin. gen

  zero in certain range — " — — " —

Grothendieck Spectral sequence

$\mathcal{A} \xrightarrow{F} \mathcal{B} \xrightarrow{G} \mathcal{C}$ abelian categories, F, G left-exact.

If F takes injectives to G-acyclics, then there exists a s.s:

$$E_2^{p,q} = (R^pG)(R^qF)(A) \Rightarrow E_\infty^n = R^n(G \circ F)(A)$$

Ex: F exact

$$\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ (R^0G)(FA) & R^1G(\cdot) & R^2G(FA) \end{array}$$

degenerates: $(R^pG)(FA) = R^p(GF)(A)$

G exact

$$\begin{array}{ccc} G(R^2F)(A) & 0 & 0 \\ G(R^1F)(A) & 0 & 0 \\ G(R^0F)(A) & 0 & 0 \end{array}$$

degenerates: $G(R^qF)(A) = R^q(GF)(A)$

Leray Spectral sequence (III Thm 1.18)

$\pi : X' \to X$ $\qquad Sh(X'_{et}) \xrightarrow{\pi_*} Sh(X_{et}) \xrightarrow{\Gamma} Ab$

π_* takes inj to inj (π^* exact) $\Rightarrow$ s.s.

$$E_2^{p,q} = H^p(X_{et}, R^q\pi_* \mathcal{F}) \Rightarrow E_\infty^{p+q} = H^{p+q}(X'_{et}, \mathcal{F})$$

$X'' \xrightarrow{\pi'} X' \xrightarrow{\pi} X$ $\qquad Sh(X''_{et}) \xrightarrow{\pi'_*} Sh(X'_{et}) \xrightarrow{\pi_*} Sh(X_{et})$

$$R^p\pi_* R^q\pi'_* \mathcal{F} \Rightarrow R^{p+q}(\pi \circ \pi')_* \mathcal{F}$$

Local-Global spectral sequence

Fix $\mathcal{F} \in Sh(X_{ét})$

$$\begin{tikzcd} Sh(X_{ét}) \arrow[r, "\mathcal{H}om(\mathcal{F},-)"] \arrow[rr, bend right, "Hom(\mathcal{F},-)"'] & Sh(X_{ét}) \arrow[r, "\Gamma(X_{ét},-)"] & Ab \end{tikzcd}$$

I injective.

$\mathcal{H}om(\mathcal{F}, I)$ not injective in general but flabby/flasque. $\left(H^i(U, \text{flabby}) = 0 \atop \forall U \to X\ ét,\ i>0\right)$

Flabby sheaves are acyclic for Γ and π_*.

We thus have S.S:

$$H^p(X_{ét}, \mathcal{E}xt^q(\mathcal{F}, \mathcal{G})) \Rightarrow Ext^{p+q}(\mathcal{F}, \mathcal{G})$$

Cohomology w/ support

$$Z \underset{\text{closed}}{\overset{i}{\hookrightarrow}} X \underset{\text{open}}{\overset{j}{\hookleftarrow}} U \qquad U = X \setminus Z$$

$$0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0 \qquad \text{SES} \qquad (*)$$

$i_* i^! \mathcal{F} \hookrightarrow \mathcal{F}$ subsheaf of sections w/ support on Z

$$\Gamma(X, i_* i^! \mathcal{F}) = \Gamma(Z, i^! \mathcal{F}) = \ker(\Gamma(X, \mathcal{F}) \to \Gamma(U, \mathcal{F}))$$

$i^!$ right adjoint of i_*, hence left exact, so:

$\mathcal{F} \longmapsto \Gamma(Z, i^! \mathcal{F})$ left exact.

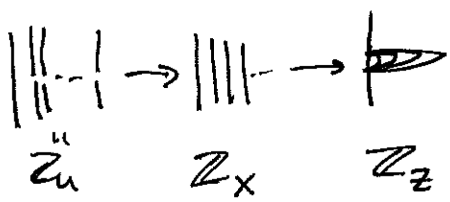
Right-derived functors:

$H^i_Z(X, \mathcal{F})$

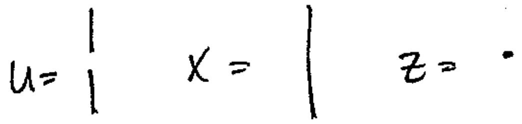
<u>Prop</u> (Milne III 1.25) $\exists$ LES (cf. <u>exactness axiom</u> of Eilenberg–Steenrod)

$$0 \to H^0_Z(X, \mathcal{F}) \to H^0(X, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_Z(X, \mathcal{F}) \to H^1(X, \mathcal{F}) \to \cdots$$

<u>pf</u>: <u>SES</u>: $0 \to \underset{\mathbb{Z}_U}{\underset{\shortparallel}{j_! j^* \mathbb{Z}}} \to \underset{\mathbb{Z}_X}{\mathbb{Z}} \to \underset{\mathbb{Z}_Z}{i_* i^* \mathbb{Z}} \to 0$



$\Rightarrow$ LES



$$\to \mathrm{Ext}^i(i_* i^* \mathbb{Z}, \mathcal{F}) \to \mathrm{Ext}^i(\mathbb{Z}, \mathcal{F}) \to \mathrm{Ext}^i(\mathbb{Z}_U, \mathcal{F}) \to$$
$$\to \mathrm{Ext}^{i+1}(i_* i^* \mathbb{Z}, \mathcal{F}) \to \cdots$$

$\mathrm{Ext}^i(\mathbb{Z}, \mathcal{F}) = H^i(X, \mathcal{F})$

$\mathrm{Ext}^i(j_! j^* \mathbb{Z}, \mathcal{F}) = H^i(U, \mathcal{F})$ b/c $\mathrm{Hom}(j_! j^* \mathbb{Z}, \mathcal{F}) = \mathrm{Hom}(j^* \mathbb{Z}, j^* \mathcal{F}) = \mathrm{Hom}_U(\mathbb{Z}, \mathcal{F}|_U) = \Gamma(U, \mathcal{F})$

$\mathrm{Ext}^i(i_* i^* \mathbb{Z}, \mathcal{F}) = H^i_Z(X, \mathcal{F})$ b/c $\mathrm{Hom}(i_* i^* \mathbb{Z}, \mathcal{F}) = \mathrm{Hom}(i^* \mathbb{Z}, i^! \mathcal{F}) = \Gamma(Z, i^! \mathcal{F}) = H^0_Z(X, \mathcal{F})$

Excision (cf. Eilenberg–Steenrod axiom)

Prop:
$$\begin{tikzcd} Z' \arrow[r, hook] \arrow[d, "\cong"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & U' \arrow[l, hook'] \arrow[d] \\ Z \arrow[r, hook, "\text{closed}"'] & X & U \arrow[l, hook', "\text{open}"] \end{tikzcd}$$
$U = X \setminus Z$, π étale, $\pi|_{Z'} : Z' \xrightarrow{\cong} Z$

Then $H^i_Z(X, \mathcal{F}) \to H^i_{Z'}(X', \pi^*\mathcal{F})$ isomorphism for all i and all $\mathcal{F} \in Sh(X)$.

pf: π^* exact and preserves injectives ($\exists$ left adjoint $\pi_!$)

Thus enough to prove that $H^0_Z(X,\mathcal{F}) \xrightarrow{\cong} H^0_{Z'}(X', \pi^*\mathcal{F})$.

$$\begin{tikzcd} 0 \arrow[r] & H^0_Z(X,\mathcal{F}) \arrow[r] \arrow[d, "\varphi"] & H^0(X,\mathcal{F}) \arrow[r] \arrow[d] & H^0(U,\mathcal{F}) \arrow[d] \\ 0 \arrow[r] & H^0_{Z'}(X',\pi^*\mathcal{F}) \arrow[r] & H^0(X', \cancel{\pi^*}\mathcal{F}) \arrow[r] & H^0(U',\mathcal{F}) \end{tikzcd}$$

Let $\gamma \in H^0_Z(X,\mathcal{F})$. If $\varphi(\gamma)=0$, then $\gamma \in H^0(X,\mathcal{F})$ goes to zero over X' and U: $\gamma|_{X'} = 0 = \gamma|_U$. Since $X' \amalg U \to X$ étale covering, this implies $\gamma = 0$. Thus φ is <u>injective</u>.

Given $\gamma' \in H^0_{Z'}(X', \pi^*\mathcal{F})$, consider it as $\gamma' \in H^0(X', \mathcal{F})$ s.th. $\gamma'|_{U'} = 0$.
Glue along $X' \amalg U \to X$. Involves $U \times_X U = U$, $U \times_X X' = U'$, $X' \times_X X'$.
$\gamma|_U := 0 = \gamma'|_{U'}$. Need to verify that $\pi_1^*\gamma = \pi_2^*\gamma$ where $\pi_1, \pi_2 : X' \times_X X' \rightrightarrows X'$.

$$\begin{array}{llll} 0 & U' \times_U U' \to X' \times_X X' & \pi_1^*\gamma', \pi_2^*\gamma' \\ & \downarrow\downarrow \qquad\quad \downarrow\downarrow & \\ 0 & U' \to X' & \gamma' \\ & \downarrow \qquad\quad \downarrow & \\ 0 & U \to X & \end{array}$$

$$\begin{tikzcd} 0 & U' \arrow[r] \arrow[d, "\Delta"'] & X' \arrow[d, "\Delta"] & \gamma' \\ 0 & U' \times_U U' \arrow[r] & X' \times_X X' & \pi_1^*\gamma', \pi_2^*\gamma' \end{tikzcd}$$

But $X' \amalg U' \times_U U' \to X' \times_X X'$ also covering so $\pi_1^*\gamma' = \pi_2^*\gamma'$
$\Rightarrow \exists \gamma$ s.th. $\gamma' = \varphi(\gamma)$.

Cohomology with compact support

Sections w/ compact support: X/k separated variety.

$$\Gamma_c(X,\mathcal{F}) = \bigcup_{\substack{Z \hookrightarrow X \\ Z \text{ proper/complete.}}} \Gamma_Z(X,\mathcal{F}) \subset \Gamma(X,\mathcal{F})$$

Ex: $k=\bar{k}$, X/k affine. $Z \hookrightarrow X$ complete $\Leftrightarrow$ Z bunch of points

$$\Gamma_c(X,\mathcal{F}) = \bigoplus_{\substack{x \in X \\ \text{closed} \\ \text{point}}} \Gamma_x(X,\mathcal{F}) = \bigoplus_{\substack{x\in X \\ \text{closed}}} \mathcal{F}_x$$

No higher cohomology ($\Gamma_c(X,-)$ exact) in this case!

Deriving $\Gamma_c(X,-)$ wrong thing.

Correct approach: Compactify $X \xhookrightarrow[\text{open}]{j} \bar{X}$. ($\exists$ by Nagata comp thm)

Define $H^p_c(X,\mathcal{F}) = H^p(\bar{X}, j_!\mathcal{F})$

Thm (later) If F torsion, then $H^p_c(X,\mathcal{F})$ independent of chosen $j: X \hookrightarrow \bar{X}$.

Prop: (Milne III 1.29)
(a) $H^0_c(X,\mathcal{F}) = \Gamma_c(X,\mathcal{F})$

(b) $H^p_c(X,-)$ δ-functor (not universal!)

(c) For any $Z \hookrightarrow X$ complete, $\exists$ morphism of δ-functors $H^p_Z(X,-) \to H^p_c(X,-)$

Rmk: Exact seq $0 \to j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F} \to 0$ gives LES

$$\dots \to H^p_c(X\setminus Z, \mathcal{F}) \to H^p_c(X,\mathcal{F}) \to H^p_c(Z,\mathcal{F}) \to \dots$$

so compact Euler char χ_c additive.