Étale cohomology #1

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Introduction

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1 Cohomology in Zariski topology

For $\mathcal{F}$ (quasi-) coherent sheaf on scheme X:

$$H^n(X, \mathcal{F})$$

$$R^i f_* \mathcal{F} \qquad f: X \to Y$$

well-behaved, e.g.

[finiteness] $\begin{cases} \cdot \ X \text{ proper}/k \Rightarrow H^p(X,\mathcal{F}) \text{ fin. dim. } k\text{-vsp} \quad \forall \mathcal{F} \text{ coh.} \\ \cdot \ \dim(X) = n \Rightarrow H^p(X,\mathcal{F}) = 0 \quad \forall p > n. \quad \forall \mathcal{F} \text{ quasi-coh.} \end{cases}$

[GAGA) $\cdot$ X proper$/\mathbb{C}$ $\Rightarrow$ $H^p(X, \mathcal{F}) \xrightarrow{\cong} H^p(X^{an}, \mathcal{F}^{an}) \quad \forall \mathcal{F}$ coh

$$\mathrm{Coh}(X) \xrightarrow{\cong} \mathrm{Coh}(X^{an}) \quad \text{equiv of cat.}$$
$$\mathcal{F} \longmapsto \mathcal{F}^{an}$$

and similarly for $R^i f_*$.

+ Serre/Grothendieck duality.

Rmk: No theory of cohomology with compact supports.

Depends on $(|X|, \mathcal{O}_X)$

2 Comparison with singular cohomology (via algebraic de Rham)

$\boxed{X \text{ proper smooth} / \mathbb{C}}$

degen. of Hodge-de Rham S.S.

$$H^n(X^{an}, \mathbb{C}) \cong H^n_{dR}(X^{an}) \otimes_{\mathbb{R}} \mathbb{C} \cong \bigoplus_{n=p+q} H^{p,q}(X^{an}) = \bigoplus H^q(X^{an}, \Omega^p_{X^{an}})$$

holomorphic diff.

GAGA: $H^q(X^{an}, \Omega^p_{X^{an}}) = H^q(X, \Omega^p_X)$

~~if X proper~~.

Can thus recover sing. coh w/ $\mathbb{C}$-coeff. from coh. of coh sheaves.
In particular, we can calculate Betti numbers.

$$b_n = \sum_{p+q=n} h_{p,q}$$

$\boxed{X \text{ smooth} / \mathbb{C}}$

$$\mathbb{C} \xrightarrow{\sim} \Omega^\bullet_{X^{an}} = [\Omega^0 \xrightarrow{d} \Omega^1 \to \cdots \to \Omega^{\dim X}]$$

so $H^n(X^{an}, \mathbb{C}) \cong \mathbb{H}^n(X^{an}, \Omega^\bullet_{X^{an}}) \cong \mathbb{H}^n(X, \Omega^\bullet_X)$

1963
Grothendieck, uses res. of sing's of a proj. emb. $X \subset \overline{X}$.

$\boxed{X \text{ sing} / \mathbb{C}}$

Deligne, Lieberman-Herrera, Hartshorne ~1971

$$H^n_{dR}(X) := \mathbb{H}^n(\hat{Z}_X, \hat{\Omega}^\bullet_Z) \cong H^n(X^{an}, \mathbb{C})$$

algebraic de Rham

$X \hookrightarrow Z$ smooth.

3 Vector bundles

vector bundle of rank n: 3 descriptions:

a) $E \longrightarrow X$ locally isomorphic to $U \times \mathbb{A}^n \to U$

b) $\mathcal{E} \in \mathrm{Coh}(X)$ —— " —— $\mathcal{O}_U^{\oplus n}$ (i.e. loc free sheaf)

c) $F \longrightarrow X$ —— " —— $U \times GL_n \to U$

More precisely, locally iso. means

i) $\exists\, X = \bigcup U_i$ open covering

ii) $E|_{U_i} \xrightarrow[\varphi_i]{\sim} U_i \times \mathbb{A}^n$

iii) $\varphi_j|_{U_i \cap U_j} \circ \varphi_i^{-1}|_{U_i\cap U_j} : (U_i \cap U_j) \times \mathbb{A}^n \longrightarrow (U_i \cap U_j) \times \mathbb{A}^n$

given by linear map $g_{ij} \in GL_n(U_i \cap U_j) = \mathrm{Hom}(U_i \cap U_j, GL_n)$

= inv. $n\times n$-matrices w/ coeff's in $\Gamma(U_i \cap U_j, \mathcal{O})$

Gives 4^{th} description

d) $X = \bigcup U_i$, $g_{ij} \in GL_n(U_i \cap U_j)$ s.th. $g_{ij} g_{jk} = g_{ik}$

i.e. vector bdls of rank n $=$ $\check{H}^1_{Zar}(X, GL_n)$ Čech cohomology.

<u>$n=1$</u>: line bundles, $\mathrm{Pic}\, X = \check{H}^1_{Zar}(X, \mathcal{O}_X^*)$ $[\, GL_1(X) = \Gamma(X, \mathcal{O}_X^*)\,]$

Exact seq: $0 \longrightarrow \mathcal{O}_X^* \longrightarrow K_X^* \longrightarrow K_X^*/\mathcal{O}_X^* \longrightarrow 0$

$\Rightarrow$ LES $H^0(K_X^*) \longrightarrow H^0(K_X^*/\mathcal{O}_X^*) \longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow 0$

$H^0(K_X^*)$: Cartier div. rat'l equiv to zero

$H^0(K_X^*/\mathcal{O}_X^*) = \mathrm{CDiv}(X)$

$H^1(X, \mathcal{O}_X^*) = \mathrm{Pic}(X)$

$\to 0$: Assuming X reduced (or has no emb. comp)

$\Rightarrow \quad \mathrm{CaCl}(X) \cong \mathrm{Pic}(X)$

4 $\mathbb{P}^1$-bundles (conic bundles)

$$\begin{array}{ccc} E = \{ z^2 = sx^2 + ty^2 \} & \hookrightarrow & X \times \mathbb{P}^2 \\ \pi \downarrow & & \\ X = \mathbb{A}^2 = \operatorname{Spec} \mathbb{C}[s,t] & & \end{array}$$

$$\begin{array}{c} \pi^{-1}(x) \quad O \cong \mathbb{P}^1 \\ \downarrow \\ x \quad \cdot \end{array}$$

Replace X w/ open s.t. fibers are smooth conics.

A smooth conic $C \hookrightarrow \mathbb{P}^2_k$ is isom to $\mathbb{P}^1$ if $C(k) \neq \emptyset$ (ie C has a k-point)

pf: projection from a k-point x gives $C \dashrightarrow \mathbb{P}^1$, extends to $C \cong \mathbb{P}^1$

So all closed fibres of π are isomorphic to $\mathbb{P}^1$ (b/c $\mathbb{C}$ alg closed)

Generic fiber: $C_\eta \hookrightarrow \mathbb{P}^2_{\mathbb{C}(s,t)}$ has no rational point!

Thus $\nexists\, U \subset X$ open s.t. $E|_U \cong U \times \mathbb{P}^1$. $\leftarrow$ Explanation: Have no section $U \to E$ giving rise to k-points of fibers

But $\exists\, U \subset X^{an}$ open and $E^{an}|_{U^{an}} \overset{\text{holo}}{\cong} U^{an} \times \mathbb{P}^1$

(use section $\begin{array}{l} x = \sqrt{t} \\ y = i\sqrt{s} \\ z = 0 \end{array}$ over some U where $\sqrt{t}, \sqrt{s}$ defined)

"Solution": Let $X' = \operatorname{Spec} \mathbb{C}[s,t,a,b]/a^2 - s,\, b^2 - t \xrightarrow[\text{ramified}]{4:1} X$ along $s=0$ and $t=0$.

$E \times_X X'$ defined by $z^2 = a^2x^2 + b^2y^2$

$$\begin{array}{c} E \times_X X' \\ s \uparrow \downarrow \\ X' \end{array} \quad \text{has section } s = \begin{cases} x = b \\ y = ia \\ z = 0 \end{cases}$$

Replace Zariski open $U \subset X$ with étale $U' \subset X' \to X$, $U' = \{ s \neq 0, t \neq 0 \}$

5 Constant coefficients

G abelian group, $H^n_{Zar}(X, G)$ is <u>badly behaved</u>

<u>Ex</u>: X <u>irreducible</u>, then every $U \subset X$ open is <u>connected</u>

$\Rightarrow$ constant sheaf $\underline{G}_X$ boring: $\underline{G}_X(U) = G \quad \forall U \subset X,\ U \neq \emptyset$.

$\Rightarrow$ $H^n_{Zar}(X, G)$ does not depend on X!

$$= H^n_{Zar}(*, G) = \begin{cases} G & \text{if } n=0 \\ 0 & \text{o/w} \end{cases}$$

<u>Ex</u>: $G \circlearrowright X$ freely. Suppose quotient $X \xrightarrow{\pi} X/G$ exists as a scheme (e.g. X quasi-proj). Then geom. fibers of π are isom to G (G-orbits of x) but $\nexists\, U \subset X/G$ open s.th. $X|_U \cong U \times G$.

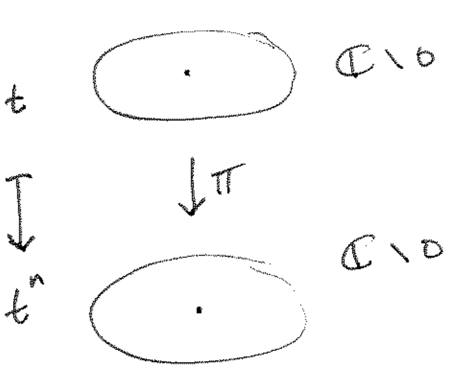
Would like: $\pi \in H^1(X/G, G)$ a <u>G-torsor</u> (principal G-bundle)

<u>Ex</u> (explicit of prev ex)

$X = \operatorname{Spec} \mathbb{C}[t, t^{-1}] \cong \mathbb{A}^1 \setminus 0$

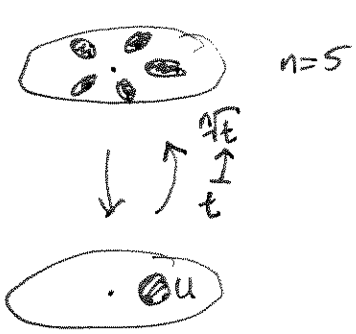
$G = \mathbb{Z}/n\mathbb{Z} \circlearrowright X \quad a.t = \zeta^a t$ where ζ prim n^{th} root of unity.

$X/G = \operatorname{Spec} \mathbb{C}[t^n, t^{-n}] \cong \mathbb{A}^1 \setminus 0$

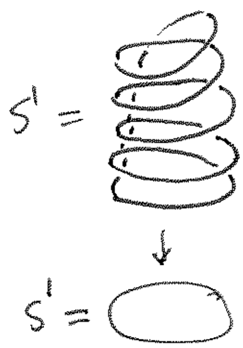


picture of cplx planes

analytic-locally: $X|_U \cong U \times G$



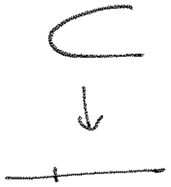
topologically homotopic to



6 More examples

Ex 1)  $k[x,y]/(y^2 - x^3 - x^2)$

irreducible but analytically-locally, singularity has 2 branches 

Ex 2)  $k[x,y]/(y^2 - x)$

$\downarrow$

$k[x]$

$x=0$

outside branch point $x=0$, analytically has 2 branches

$x=1$

Ex 1) branches given by $y = \pm x\sqrt{x+1} = \pm x\left(1 + \frac{x}{2} - \frac{x^2}{8} + \dots\right)$ nbhd of $x=y=0$

Ex 2) —— " —— $y = \pm\sqrt{x} = \pm\sqrt{1+(x-1)} = \pm\left(1 + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \dots\right)$ nbhd of $x=1$

Topology closely connected to concept of <u>local rings</u>

$$\mathcal{O}_{X,x} \subset \mathcal{O}_{X^{an},x} \subset \hat{\mathcal{O}}_{X,x}$$

usual local ring — convergent power series — formal power series

Zariski top — analytic top

<u>Étale topology</u>: $\mathcal{O}_{X^h,x}$ <u>henselian</u> local ring $\quad \mathcal{O}_{X,x} \subset \mathcal{O}_{X^h,x} \subset \mathcal{O}_{X^{an},x}$

"algebraic power series"

e.g. $\sqrt{x+1}$, $\sqrt{x}$

Ex 1) henselian local ring at sing is not an integral domain: has 2 irr. components

$y = \pm x\sqrt{x+1}$

Ex 2) $\exists$ 2 disjoint sections over henselian local ring at $x=1$.

$y = \sqrt{x}$, $y = -\sqrt{x}$

7 Implicit function theorem

$$\begin{array}{l} x \in X = \operatorname{Spec} A[x_1, \dots, x_n]/(f_1, \dots, f_n) \\ \quad p \downarrow \\ \quad\; Y = \operatorname{Spec} A \end{array}$$

If $\left(\frac{\partial f_i}{\partial x_j}\right)$ invertible at x, then $\exists\, U \subset Y^{an}$ analytic open, $U \to X$ section through x

(implicit fcn thm).

That is, p is a local isomorphism at x in analytic topology.

Not true in Zariski topology.

Solution: Add $X \to Y$ as an "open subset" — an étale morphism.

To be precise: $d = \det\left(\frac{\partial f_i}{\partial x_j}\right)$ polynomial in $A[x_1, \dots, x_n]$. After inverting d implicit fcn thm holds everywhere. The morphism $\underset{\{d \neq 0\}}{\underset{\shortparallel}{V}} \subset X \to Y$ is étale.

"Def": $X \to Y$ étale if $X^{an} \to Y^{an}$ local isomorphism.

Rmk: $X \to Y$ finite étale $\Leftrightarrow$ $X^{an} \to Y^{an}$ finite covering space

(e.g. $\mathbb{A}^1 \setminus 0 \to \mathbb{A}^1 \setminus 0$
$t \mapsto t^n$)

8 Why could this possibly work? [Danilov Ch 3 §4]

Riemann's existence thm / "Gr-R"

$X/\mathbb{C}$. $\quad \mathrm{F\acute{E}t}(X) \xrightarrow{\sim} \mathrm{F\acute{E}t}(X^{an}) =$ finite covering spaces of X^{an}

$\quad (Z \to X) \mapsto (Z^{an} \to X^{an})$

pf: X proper. Then follows from GAGA for coh sheaves

X gen. Use res. of sing's (or Grauert-Remmert's thm)

Fundamental group $\pi_1(X,x)$ can be defined via

1) loops
2) covering spaces

RET suggests defining $\pi_1^{\acute{e}t}(X,x)$ using finite ét

$\Rightarrow \pi_1^{\acute{e}t}(X,x) = \pi_1(X^{an}, x)^\wedge \leftarrow$ pro-finite completion.

For $H^n(X, G)$, $n>1$:

Thm (Artin) $X/\mathbb{C}$ smooth. every $x \in X$ has an open nbhd U s.t.

$\pi_n(U^{an}) = 0 \quad \forall n > 1$. $\quad$ (a "nice nbhd" = "bon voisinage")

Thus a mixture of open coverings and finite étale covers could give a good cohomology theory.

Ex: X curve of genus g. Open subsets $\leftrightarrow$ adding punctures

w/o punctures (X complete) $\pi_2(X) \neq 0$ but w/ $m \geq 1$ punctures

$X^{an} \sim$ wedge of $2g+m-1$ circles $\Rightarrow \pi_2(X) = 0$.

$g=1$, $m=1$

9 Étale cohomology

For A finite (torsion) abelian group (order prime to char(k))

$H^n_{ét}(X, A)$

$R^i f_* A$

well-behaved. If X/k finite type

finiteness
- $H^n_{ét}(X, A)$ finite ab. grp $\forall n$
- $H^n_{ét}(X, A) = 0 \quad \forall n > 2\dim X$

GAGA
- $H^n_{ét}(X, A) = H^n(X^{an}, A)$ if $X/\mathbb{C}$ finite type

+ Poincaré duality, Lefschetz trace formula, coh. w/ compact support

Also $\pi_1^{ét}$ well-behaved.

Relative setting important: $f: X \longrightarrow Y$

$R^i f_* A$ not (locally) constant[†] but constructible. (= finite stalks)

Many proofs via curve fibrations.

† unless f proper + smooth.

10 Why should we care about Étale cohomology?

char 0

- if X defined over number field K $\Rightarrow$ $Gal(K) \circlearrowright H^*(X, \mathbb{C})$
- powerful char p methods (Frob.) $\rightsquigarrow$ deep results in char 0
  point-counting (e.g. decomposition theorem)
- purely algebraic proofs

char p

- only choice besides Crystalline coh, rigid coh and their variants.
  (and needs all of them)
- Weil conjectures
- arithmetic geometry

Moreover, étale topology has many other applications.