LECTURE 5: Applications

"Local applications"

5.1–5.2 Sumihiro, Luna, $\mathbb{L}_{\mathcal{X}}$, $\mathcal{X}_x^\wedge$, $\mathcal{X}_x^h$,

5.3 { res prop (gms, group schemes)
good vs adequate

Global applications

5.4 { Compact generation of derived categories
Cohomology and base change

5.5 Algebraicity of moduli stacks (Coh, Quot, Hilb, Map, ...)

5.6 Białynicki-Birula decompositions of DM-stacks

5.7 Partial Kirwan desingularization for stacks

5.8 Existence of good moduli spaces.

5.1 Application: Equivariant geometry (Sumihiro + Luna)

Thm "Sumihiro": G affine flat $\circlearrowright X$ alg space, $x \in |X|$, G_x lin red.
(everything f.p. $/S$ and $\kappa(x)/\kappa(s)$ finite, e.g. x closed)

[AHR1, 4.4]
[AHR2, 17.1]

$$\begin{tikzcd} \exists\ U = \operatorname{Spec} A \arrow[r] & X \\ w \arrow[r, mapsto] \arrow[u, phantom, "\in" description] & x \arrow[u, phantom, "\in" description] \end{tikzcd} \quad G\text{-equi étale, } G_w \cong G_x.$$

pf:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ \boxed{\mathcal{W} \text{ lin fund}} \arrow[r, "\text{étale}"] & {[X/G]} \arrow[r] & BG \end{tikzcd}$$

Thm LS.

Refinement (a) in Thm LS $\Rightarrow$ $\mathcal{W} \to BG$ affine
$\Rightarrow$ U affine. ∎

Thm "Sumihiro" T torus $\circlearrowright X$ DM-stack, $x \in |X|$. Then after reparametrization of T (i.e. repl X with $X_{\langle n \rangle}$ where $\lambda x = \lambda^n x$)
$X_{\langle n \rangle} = X$ as ~~stacks~~

[AHR1, 4.1 + 4.3]

$\exists\ u \in \operatorname{Spec} A \to X_{\langle n \rangle}$ T-equi $T_u \cong T_x$

pf: similar.

Thm "Luna": G affine smooth $\circlearrowright X$ alg space, $x \in |X|$, G_x lin red.

[AHR1, 4.5]
[AHR2, 17.4]

$\exists\ U = \operatorname{Spec} A \to X$ unramified G_x-equi
$G_x \circlearrowright U$ s.th. $[U/G_x] \to [X/G]$ étale.

(orig Luna: X aff sch. $U \to X$ loc closed im /)

pf:

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d] & S \arrow[d] \\ \mathcal{W} \arrow[r, dashed, "\text{unramif}"] \arrow[dr, "\text{étale}"', "\text{Thm LS}"] & {[X/G_x]} \arrow[r] \arrow[d, "\text{smooth affine}"] & BG_x \\ & {[X/G]} & \end{tikzcd}$$

(1) $\mathcal{W} \xrightarrow{\text{ét}} [X/G]$ (Thm LS)
(2) lift $\mathcal{W}_0 \to [X/G_x]$ to $\mathcal{W} \to [X/G_x]$
$\mathcal{W}_0 = BG_x$ (def thy + CP + AA) □

5.2 Application: Local structure of stacks.

$x \in \mathcal{X}$, G_x lin red (and $\mathcal{X}$ aff. stabs)

(1) $\exists\ \mathcal{W} = [\mathrm{Spec}(A)/G_x] \xrightarrow[f]{\text{ét}} \mathcal{X}$ "Étale-locally smooth embeddings"

$$\begin{array}{c} \mathcal{W} \\ \downarrow i \\ \mathcal{Y} = [\underset{V}{\text{smooth}}/G_x] \end{array}$$

$$\Rightarrow \tau^{\geq -1} f^* L_{\mathcal{X}} = [\underset{-1}{I/I^2} \to \underset{0}{\Omega_V|_{\mathcal{W}}} \to \underset{1}{\mathfrak{g}_x^\vee}]$$

(2) $\exists\ \mathcal{X}_x^h = \mathcal{W}_w^h = \mathcal{W} \times_W W_w^h$

(3) $\exists\ \mathcal{X}_x^\wedge = \mathcal{W}_w^\wedge = \mathcal{W} \times_W W_w^\wedge$

(4) Given $(\mathcal{X}, x)$, $(\mathcal{Y}, y)$ $\quad \mathcal{X}_x^\wedge \cong \mathcal{Y}_y^\wedge \iff$

$$\begin{array}{ccc} & (\mathcal{W}, w) & \\ {}^{\text{ét}}\swarrow & & \searrow^{\text{ét}} \\ (\mathcal{X}, x) & & (\mathcal{Y}, y) \end{array}$$

(5) If $\mathcal{X}$ has lin red stabs at <u>all</u> points (almost never happens except when all of <u>mult type</u>)

$\exists \coprod_i \mathcal{W}_i \to \mathcal{X}$ <u>Nisnevich</u> cover where $\mathcal{W}_i = [\mathrm{aff}/\text{lin red}]$

(used by Hoyois–Krishna in K-theory for stacks)

5.3 Application: Resolution property and gms/groups

Thm: $\mathcal{X} \xrightarrow{\text{gms}} X$, $\Delta_{\mathcal{X}}$ separated. Then $\exists$

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\ X' \arrow[r, "\text{ét}"] & X \end{tikzcd}$$

s.th. $\mathcal{X}'$ has res. prop.

even Nisnevich

In particular $\Delta_{\mathcal{X}/X}$ affine.

(immediate consequence of Thm LS w/ refinements (S) + (C))

Cor: $G \to S$ separated flat lin red $\Rightarrow \exists S' \to S$ étale (even Nis.) s.th. $G \times_S S' \overset{\text{closed}}{\hookrightarrow} GL_{n,S'}$. In part, G affine.

Previously, Thm wasn't even known when $X = \operatorname{Spec} k$!.

Thm: [AHR2, §16] $G \to S$ separated flat f.p. ~~lin red~~ group scheme. TFAE

(1) $G \to S$ lin red

(2) $\exists\ 1 \to G^{\circ}_{\text{red}} \to G \to G/G^{\circ}_{\text{red}} \to 1$ (and $G^{\circ}_{\text{red}} \triangleleft G$ canonical)

G°_{red}: smooth, connected, lin red

G/G°_{red}: finite, flat, tame

Thm: [AHR2, 9.3] $\mathcal{X} \xrightarrow{\text{ams}} X$ <u>adequate</u> mod space of finite type. If $x \in |\mathcal{X}|$ closed, G_x <u>lin red</u> then $\mathcal{X} \to X$ <u>good</u> in nbhd of x.

5.4 Application: Derived categories (compact generation)

Thm [AHR1 4.27] [AHR3 ...] $\mathcal{X}$ qcqs algebraic stack, with affine diagonal. If $(G_x^0)_{red}$ lin red. $\forall$ closed point $x \in |\mathcal{X}|$ then $D_{qc}(\mathcal{X})$ compactly generated (by perfect complexes)

} e.g. $\mathcal{X}$ admits gms

pf:

$$\begin{tikzcd} W_0 = B(G_x^0)_{red} \arrow[r] \arrow[d, hook] & \mathcal{X}_0 \arrow[d, hook] \\ W \arrow[r, "p"] & \mathcal{X} \end{tikzcd}$$

finite syntomic

qfin flat (synth)

lin fund (res prop.)

Thm [HR17]: "$D_{qc}(W)$ comp gen $\Rightarrow D_{qc}(\mathcal{X})$ comp. gen'd" $\square$

Precise condition: in [HR17]

W "Thomason property", also étale-locally $\Rightarrow \mathcal{X}$ has "Thomason property"

Def: $\mathcal{X}$ Thomason prop if
- $D_{qc}(\mathcal{X})$ cpt gen.
- $\forall \mathcal{U} \subset \mathcal{X}$ qcqs $\exists P$ perfect w/ $\mathrm{Supp}(P) = \mathcal{X} \setminus \mathcal{U}$.

Fact: res prop $\Rightarrow$ Thomason prop.

Cor: [AHR1 4.36] "cohomology and base change" holds for ~~such~~ $\mathcal{X}$ noetherian w/ gms.

5.5 Application: Algebraicity

$$\begin{tikzcd} \mathcal{X} \arrow[d, "p", "\text{gms}"'] \\ S \end{tikzcd}$$ (morally should be enough with $\begin{tikzcd} \mathcal{X} \arrow[d, "\text{gms}"] \\ X \arrow[d, "\text{proper}"] \\ S \end{tikzcd}$ but don't know this)

$Coh(\mathcal{X}/S)$, $Quot(\mathcal{X}/S, \mathcal{F})$, $Hilb(\mathcal{X}/S)$ are algebraic

If p flat and $\mathcal{Y} \to S$ arbitrary:

$Map_S(\mathcal{X}, \mathcal{Y})$ algebraic stack

pf: Artin's axioms.
- Thm CP $\Rightarrow$ effectivity
- $D_{qc}(\mathcal{X})$ cpt gen'd $\Rightarrow$ coh & b.c $\Rightarrow$ def thy well-behaved.

Cor: $G \circlearrowright X, Y / S$, $X /\!/ G \xrightarrow{\cong} S$, $Map^G(X, Y)$ algebraic space.
$X \to S$ flat

5.6 Application: BB-decompositions

$\mathbb{G}_m \curvearrowright X$ separated algebraic space or DM-stack over field k.

$X^{\mathbb{G}_m} = \mathrm{Map}^{\mathbb{G}_m}(\mathrm{Spec}\, k, X) \hookrightarrow X$ closed substack (separ. $\mathbb{G}_m$ if X DM-stack)

$$\begin{tikzcd} X^+ = \mathrm{Map}^{\mathbb{G}_m}(\mathbb{A}^1, X) \arrow[r, hook, "ev_1"] \arrow[dr, "ev_0"'] & X \text{ monomorphism} \\ & X^{\mathbb{G}_m} \arrow[u, hook] \end{tikzcd}$$

ev_0 always <u>affine</u>.

<u>Thm</u> (BB for DM-stacks) Let $X^{\mathbb{G}_m} = \coprod \mathcal{F}_i$ (conn components)
and $X^+ = \coprod \mathcal{X}_i$ (inv. images of $\mathcal{F}_i$)

(1) If X proper $X^+ \hookrightarrow X$ surjective.
(2) if X smooth then $\mathcal{F}_i$ smooth and $\mathcal{X}_i \to \mathcal{F}_i$ affine bundle
(3) if X smooth and X_{cms} a <u>scheme</u> then $\mathcal{X}_i \hookrightarrow X$ <u>loc closed</u> imm
or X normal and X_{cms} qproj scheme

<u>main idea of pf</u>: Reduce to X affine using Sumihiro-generalization.

5.7 Application: Partial Kirwan desing / destackification

Recall GIT-connection: G lin red $\circlearrowright X$ qproj, $\mathcal{L}$ G-linear ample l.b.

$$\Rightarrow \begin{tikzcd} & X^{ss} \arrow[d] \\ {[X^s/G]} = \mathcal{X}^s \arrow[d] \arrow[r, phantom, "\subset"] & {[X^{ss}/G]} = \mathcal{X} \arrow[d, "gms"] \\ X^s/G \arrow[r, phantom, "\subset"] & X^{ss}/\!\!/G \end{tikzcd}$$

Thm [ER18] $\mathcal{X} \xrightarrow{\pi} X$ gms (f.t./k) s.th. π cms over dense open. ($\mathcal{X}^s$)

$\exists$ canonical seq $\mathcal{X}_n \to \mathcal{X}_{n-1} \to \cdots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{X}$

of saturated blow-ups (qproj birational) s.th.

(1) $\exists$ gms $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X$ (usual blow-ups)

(2) maps iso over $\mathcal{X}^s$.

(3) $\mathcal{X}_n \to X_n$ cms. (all stab of $\mathcal{X}_n$ has same dimension: "$s = ss$")

(4) If $\mathcal{X}$ smooth, then all $\mathcal{X}_i$ smooth.

Combining w/ destackification (Bergh '17) if $\mathcal{X}$ smooth

$$\exists \begin{tikzcd} \mathcal{X}'_n \arrow[r] \arrow[d, "\substack{\text{gerbe}\\ \text{root stacks}}"] & \mathcal{X}_n \\ X'_n \text{ smooth} & \end{tikzcd} \quad \text{blow-ups root stacks}$$

Algorithm: $\mathcal{X}_{n+1} = Bl'_{\mathcal{Z}_n}\mathcal{X}_n \ \forall n$ where $\mathcal{Z}_n \hookrightarrow \mathcal{X}_n$ locus of max stab (smooth if $\mathcal{X}_n$ smooth)

$$\text{sat. blow-up} \quad \begin{tikzcd} Bl'_{\mathcal{Z}_n}\mathcal{X}_n \arrow[r, phantom, "\subset"] \arrow[d, "gms"] \arrow[rr, bend left, "qproj"] & Bl_{\mathcal{Z}_n}\mathcal{X}_n \arrow[r] \arrow[dl, dashed, "\text{rat map}"] & \mathcal{X}_n \arrow[d, "gms"] \\ Bl_{Z_n}X_n \arrow[rr, "proj"'] & & X_n \end{tikzcd}$$

Use local str. thm to prove that
- $\mathcal{Z}_n$ smooth
- max dim stab$(\mathcal{X}_{n+1})$ < max dim stab$(\mathcal{X}_n)$.

5.8 Application: Existence of good moduli spaces

Thm [AHLH 18] $\mathcal{X}$ affine diag, f.t $/S$ noetherian $\mathbb{Q}$-scheme.

Then $\mathcal{X}$ admits a gms $\mathcal{X} \to X$ w/ X separated iff

(1) $\mathcal{X}$ Θ-reductive

(2) $\mathcal{X}$ S-complete. (S = Seshadri)

(also precise cond in pos. char. and w/o X separated)

Def: $\mathcal{X}$ Θ-reductive if $\forall R$ DVR:

$$\begin{tikzcd} \Theta_R \setminus 0 \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{X} \arrow[d] \\ \Theta_R \arrow[r] \arrow[ur, dashed, "\exists !"] & S \end{tikzcd}$$

$$\Theta_R = [\mathbb{A}^1/\mathbb{G}_m] \times \operatorname{Spec} R$$

$$0 = (0, 0)$$

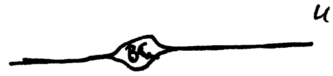
$$\Theta_R \setminus 0 = [\mathbb{A}^1_K/\mathbb{G}_m] \cup_{\operatorname{Spec} K} \operatorname{Spec} R$$

$\mathcal{X}$ S-complete if $\forall R$ DVR:

$$\begin{tikzcd} \overline{ST}_R \setminus 0 \arrow[r] \arrow[d, phantom, "\cap"] & \mathcal{X} \arrow[d] \\ \overline{ST}_R \arrow[r] \arrow[ur, dashed, "\exists !"] & S \end{tikzcd}$$

$$\overline{ST}_R = [\operatorname{Spec} R[s,t]/(st-\pi) / \mathbb{G}_m]$$

weights $+1, -1$

$=$ 

$$\overline{ST}_R \setminus 0 = \operatorname{Spec} R \cup_{\operatorname{Spec} K} \operatorname{Spec} R$$