LECTURE 4: Tannaka duality, Artin approximation and equi. Artin algebraization

4.1 Tannaka duality (Thm TD)

4.2 Effectivity (Thm E)

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4.3 Proofs of Step 2 and 3 of local str. thm

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4.8 (Equivariant) Artin algebraization (Thm EAA)

4.9 Proof of Step 4 (general case)

4.1 Tannaka duality

$f : \mathcal{X} \longrightarrow \mathcal{Y}$ induces $f^* : QCoh(\mathcal{Y}) \longrightarrow QCoh(\mathcal{X})$

(also $f^* : Coh(\mathcal{Y}) \longrightarrow Coh(\mathcal{X})$, if $\mathcal{X}, \mathcal{Y}$ noeth)

s.th. (0) f^* additive

(i) $f^* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$

(ii) $f^*(\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{Y}}} \mathcal{G}) = f^*\mathcal{F} \otimes_{\mathcal{O}_{\mathcal{X}}} f^*\mathcal{G}$

$\Big\}$ (symm.) monoidal functor

(iii) f^* cocontinuous (preserves colimits), i.e., right exact and $f^*(\bigoplus \mathcal{F}_\alpha) = \bigoplus (f^* \mathcal{F}_\alpha)$

(f^* right-exact for Coh version)

Theorem TD: [HR19, Thm 1.1]

$\mathcal{X}$ excellent alg stack
$\mathcal{Y}$ noetherian alg stack w/ affine stabilizers (no other sep. assump!)

Variant:
$\mathcal{X}$ noetherian
$\mathcal{Y}$ noetherian, $\Delta_{\mathcal{Y}}$ qaff

Then:

$$\mathrm{Map}(\mathcal{X}, \mathcal{Y}) \longrightarrow \mathrm{Hom}_{r\otimes, \simeq}(Coh(\mathcal{Y}), Coh(\mathcal{X}))$$
$$f \longmapsto f^*$$

equivalence of groupoids.

$r\otimes$ = objects are right-exact tensor functor (additive + monoidal)

$\simeq$ = morphisms are natural isomorphisms

Corollary: If $(\mathcal{X}, \mathcal{X}_0)$ excellent complete pair w/ aff stab.
then $\mathcal{X} = \varinjlim_n \mathcal{X}^{[n]}$ in category of noeth alg stacks w/ aff stab.

proof:

$$\mathrm{Map}(\mathcal{X}, \mathcal{Y}) \overset{TD}{=} \mathrm{Hom}_{r\otimes, \simeq}(Coh(\mathcal{Y}), Coh(\mathcal{X}))$$
$$\overset{CP}{=} \mathrm{Hom}_{r\otimes, \simeq}(Coh(\mathcal{Y}), \varprojlim Coh(\mathcal{X}^{[i]}))$$
$$= \varprojlim \mathrm{Hom}_{r\otimes, \simeq}(Coh(\mathcal{Y}), Coh(\mathcal{X}^{[i]}))$$
$$\overset{TD}{=} \varprojlim \mathrm{Map}(\mathcal{X}^{[i]}, \mathcal{Y})$$

Remarks: • TD $\Rightarrow$ $\mathrm{AlgStacks}_{\text{exc, aff stab}} \xhookrightarrow[\text{full faithful}]{} \mathrm{ATC}$ = Grothendieck abelian tensor categories

$$\mathcal{X} \longmapsto \mathrm{QCoh}(\mathcal{X})$$

- aff stab crucial: $\mathrm{QCoh}(BA) = \mathrm{QCoh}(*)$ if $A \to *$ abelian variety.
- $\otimes$ crucial: $\mathrm{QCoh}(B\mu_2) \cong \mathrm{QCoh}(* \amalg *)$ as <u>abelian categories</u>

  w/ $\otimes$: LHS = $(\mathbb{Z}/2\mathbb{Z})$-graded vsp $V_0 \oplus V_1$
  RHS = (V, W) V, W v.sp.

- Derived version w/ D_{qc} instead of QCoh (Bhatt – Halpern-Leistner) for quasi-affine diagonal.

<u>Outline of proof:</u>

(1) $\mathcal{Y}$ aff stab $\Longleftrightarrow$ $\exists$ stratification w/ strata $\mathcal{Y}_i = [\text{qaff}/GL_n]$

(2) TD holds for $\mathcal{Y} = \mathcal{Y}_i$ (easy: use that vector bundles are preserved by tensor functors b/c v.b = dualizable)

(3) <u>Main lemma</u>: TD holds for $\mathcal{Y} = \mathcal{Y}_i^{[n]}$ $\forall n$.

(4) Induction on # strata $\Rightarrow$ WLOG $\mathcal{Y}_1 \xhookrightarrow[\text{closed}]{} \mathcal{Y} \underset{\text{open}}{\supseteq} \mathcal{Y}_2 = \mathcal{Y} \setminus \mathcal{Y}_1$

"pullback" to $\mathcal{X}$: $\mathcal{X}_1 \hookrightarrow \mathcal{X} \supseteq \mathcal{X}_2$.

(5) <u>Tensor localization</u> gives tensor functor $\mathrm{Coh}(\mathcal{Y}_2) \to \mathrm{Coh}(\mathcal{X}_2)$ $\overset{\text{induction}}{\rightsquigarrow}$ $\mathcal{X}_2 \to \mathcal{Y}_2$

Step (3) gives $\mathcal{X}_1^{[n]} \to \mathcal{Y}_1^{[n]}$

(6) <u>Formal gluings</u> "Mayer-Vietoris" gives pushout square (if $\mathcal{X}$ <u>excellent</u>) (WLOG $\mathcal{X} = X$ affine by descent)

$$\begin{tikzcd}
X_2 \times_X \hat{X} \arrow[r] \arrow[d, phantom, "\cap"] \arrow[dr, phantom, "\square"] & X_2 \arrow[d, phantom, "\cap"] \arrow[dr] & \\
\hat{X} \arrow[r] & X \arrow[r, dashed] & \mathcal{Y} \\
X_1^{[n]} \arrow[u, hook] \arrow[r, equal] & X_1^{[n]} \arrow[u, hook] \arrow[ur] &
\end{tikzcd}$$

$\Rightarrow$ $X \xrightarrow{f} \mathcal{Y}$.

4.2 Effectivity

Def: An <u>adic sequence</u> of stacks is a diagram of closed immersions
[AHR2, 1.8]

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \hookrightarrow \cdots$$

s.th. $\forall i<j$, $\mathcal{X}_i = i^{th}$ inf nbhd of $\mathcal{X}_0$ in $\mathcal{X}_j$.

Ex: If $\mathcal{X}_0 \hookrightarrow \mathcal{X}$, then $\mathcal{X}_*^{[0]} \hookrightarrow \mathcal{X}^{[1]} \hookrightarrow \cdots$ adic seq.

Def: A <u>coherent completion</u> of an adic seq $\{\mathcal{X}_n\}$ is a stack $\hat{\mathcal{X}}$ s.th.
[AHR2, 1.9]

(1) $\exists\ \mathcal{X}_i \hookrightarrow \mathcal{X}$ comp. closed immersions.
(2) $\hat{\mathcal{X}}$ noeth, w/ affine diag
(3) $(\hat{\mathcal{X}}, \mathcal{X}_0)$ complete.

Rmk: TD $\Rightarrow$ $\hat{\mathcal{X}} = \varinjlim \mathcal{X}_i$ in category of noeth stacks w/ qaff diag (or exc stacks w/ aff stab)

Ex: $\mathcal{X}_x^\wedge :=$ coh. completion of $\{\mathcal{X}_x^{[n]}\}$.

Thm E: $\{\mathcal{X}_i\}$ adic seq of noeth stacks.
[AHR2, 1.10] $\mathcal{X}_0$ lin fund $\Rightarrow$ $\hat{\mathcal{X}}$ exists and is lin fund.

Rmk: Have seen that $\mathcal{X}_i$ lin. fund. Thus have:

$$\begin{array}{ccccc}
\mathcal{X}_0 & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \\
\downarrow & & \downarrow & & \cdots \\
X_0 & \hookrightarrow & X_1 & \hookrightarrow & \\
\| & & \| & & \\
\operatorname{Spec} A_0 & & \operatorname{Spec} A_1 & &
\end{array}$$

Problem: $X_0 \hookrightarrow X_1 \hookrightarrow \cdots$ <u>not</u> adic.

we will have that $\hat{X} = \varprojlim X_n$ gms of $\hat{\mathcal{X}}$ but a priori not even noetherian!

4.3 Proofs of Steps 2 and 3

$$\begin{array}{ccc} W_0 & \xrightarrow{f_0} & \mathfrak{X}_0 \\ \text{lin fund} & & \downarrow \\ & & \mathfrak{X} \end{array} \quad \text{étale/smooth/sync.}$$

Step 1 gives adic seq $\{W_i\}$ and $W_i \xrightarrow{f_i} \mathfrak{X}^{[i]}$

Step 2: Thm E $\Rightarrow \exists\, \hat{W} = \varinjlim W_i$ lin fund, $(\hat{W}, W_0)$ coh. complete.

Step 3: Thm TD $\Rightarrow \exists\, \hat{f} : \hat{W} \longrightarrow \mathfrak{X}$ extending the f_i.

$\hat{f}$ flat (loc crit of flatness) but not necc of finite type

4.4 Proof of effectivity when $\mathcal{X}/\mathrm{Spec}\, k$ and $X_0 = \mathrm{Spec}\, k$, $k = \bar{k}$. [AHR1, pf Thm 1.1]

Let $~~\mathcal{I}/\mathcal{I}~~ \mathcal{I}/\mathcal{I}^2$ = conormal sheaf of $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1$.

$\mathcal{I}/\mathcal{I}^2 \in \mathrm{Coh}(\mathcal{X}_0) = VB(BG_x) =$ fin dim G_x-rep. V

Let $\mathcal{N} = \mathrm{Spec}_{BG_x} \mathrm{Sym}(\mathcal{I}/\mathcal{I}^2) = [\check{V}/G_x]$ <u>normal stack</u>. (smooth)

<u>Lemma 1</u>: All square-zero extensions of $\mathcal{X}_0$ w/ ideal $\mathcal{J}$ are isomorphic (all trivial)

<u>pf</u>: $\mathrm{Exal}_k(BG_x, \mathcal{J}) = \mathrm{Ext}^1_{BG_x}(L_{BG_x/k}, \mathcal{J}) = H^1(BG_x, \underbrace{L^\vee_{BG_x/k} \otimes \mathcal{J}}_{\substack{\text{Tor-amp } [-1,0] \\ \text{b/c } BG_x \to \mathrm{Spec}\, k \\ \text{smooth}}}) = 0$

$$\begin{tikzcd}
BG_x = \mathcal{X}_0 \arrow[r, hook] \arrow[d, equal] & \mathcal{X}_1 \arrow[d, equal] \arrow[dr, hook, "\phi_1"] & \\
BG_x = \mathcal{N}^{[0]} \arrow[r, hook] & \mathcal{N}^{[1]} \arrow[r, hook] & \mathcal{N}
\end{tikzcd}$$

<u>Lemma 2</u>: We can lift $\mathcal{X}_n \xrightarrow{\phi_n} \mathcal{N}$ to $\mathcal{X}_{n+1} \xrightarrow{\phi_{n+1}} \mathcal{N}$ $\forall n$.

<u>pf</u>: obs $\in \mathrm{Ext}^1_{\mathcal{X}_n}(\phi_n^* L_{\mathcal{N}/k}, \mathcal{J}) = H^1(\mathcal{X}_n, \underbrace{\phi_n^* L^\vee_{\mathcal{N}/k} \otimes \mathcal{J}}_{\substack{\text{Tor-amp } [-1,0] \\ \text{b/c } \mathcal{N} \to \mathrm{Spec}\, k \ \text{smooth}}}) = 0$.

<u>Lemma 3</u>: ϕ_1 closed immersion $\Rightarrow$ ϕ_n closed imm $\forall n$.

[AHR1, Prop A.8 (1)]
[AHR2, Lem 6.3]

Follows from:

<u>Lemma 4</u>: $g: A \to B$ s.th. $A \to B \to B/I$ surj and $I^n = 0$. Then g surjective.

<u>pf</u>: Use Nakayama (for nilpotent ideals) twice: first on B to show $\ker(g)B = I$.
Then on A to show that $A \to B$ surj.

proof of effectivity over field $\bar{k}$ continued

Since $\{\mathcal{X}_i\}$ adic and $\phi_n : \mathcal{X}_n \hookrightarrow \mathcal{N}$ closed imm, we get:

$$\begin{tikzcd}
\mathcal{X}_0 \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \mathcal{X}_1 \arrow[r, hook] \arrow[d, equal] \arrow[dr, phantom, "\square"] & \mathcal{X}_2 \arrow[r, hook] \arrow[d, hook, "\text{closed imm}"] \arrow[dr, phantom, "\square"] & \mathcal{X}_3 \arrow[r, hook] \arrow[d, hook, "\text{closed imm}"] & \cdots \\
\mathcal{N}^{[0]} \arrow[r, hook] & \mathcal{N}^{[1]} \arrow[r, hook] & \mathcal{N}^{[2]} \arrow[r, hook] & \mathcal{N}^{[3]} \arrow[r, hook] & \cdots
\end{tikzcd}$$

$\Rightarrow \quad \{\mathcal{O}_{\mathcal{X}_n}\} \in \varprojlim \mathrm{Coh}(\mathcal{N}^{[n]})$

Def: $N := \mathrm{Spec}(\Gamma(\mathcal{N}, \mathcal{O}_{\mathcal{N}}))$ i.e. gms of $\mathcal{N}$.

Def: $\hat{\mathcal{N}} := \mathcal{N} \times_N N_0^\wedge$

$$\begin{tikzcd}
\hat{\mathcal{N}} \arrow[r] \arrow[d, "\text{gms}"'] \arrow[dr, phantom, "\square"] & \mathcal{N} \arrow[d, "\text{gms}"] \\
N_0^\wedge \arrow[r] & N
\end{tikzcd}$$

Thm CP $\Rightarrow$ $(\hat{\mathcal{N}}, 0)$ complete, i.e. $\mathrm{Coh}(\hat{\mathcal{N}}) = \varprojlim \mathrm{Coh}(\mathcal{N}^{[n]})$

$\Rightarrow \quad \mathcal{O}_{\hat{\mathcal{X}}} \in \mathrm{Coh}(\hat{\mathcal{N}})$ effectivizing $\{\mathcal{O}_{\mathcal{X}_n}\}$.

This is an $\mathcal{O}_{\hat{\mathcal{N}}}$-alg equipped w/ $\mathcal{O}_{\hat{\mathcal{N}}} \twoheadrightarrow \mathcal{O}_{\hat{\mathcal{X}}}$, i.e.

$\hat{\mathcal{X}} \overset{\phi}{\hookrightarrow} \hat{\mathcal{N}}$ closed imm. s.th.
$$\begin{tikzcd}
\mathcal{X}_n \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \hat{\mathcal{X}} \arrow[d, hook] \\
\mathcal{N}^{[n]} \arrow[r, hook] & \hat{\mathcal{N}}
\end{tikzcd} \quad \forall n.$$

- Verify (Exc) that $(\hat{\mathcal{X}}, \mathcal{X}_0)$ coh. complete
- By construction $\hat{\mathcal{X}} \hookrightarrow \hat{\mathcal{N}} \to \mathcal{N} \to BG_x$ affine, i.e., linearly fundamental $\square$

4.5 Proof of Thm E (char 0)

[AHR2, §6]

$$\begin{array}{ccccl} \mathcal{X}_0 & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow \cdots & \text{adic seq of (lin) fund stacks } /\mathbb{Q} \text{ (noetherian)} \\ {\scriptstyle \pi_0}\downarrow & & \downarrow & & \\ X_0 & \hookrightarrow & X_1 & \hookrightarrow & \text{their gms, } \underline{\text{not}} \text{ adic.} \end{array}$$

$X_n = \operatorname{Spec} A_n$

$X = \operatorname{Spec} A$, $A = \varprojlim A_n$

$I_{n+1} = \ker(A \twoheadrightarrow A_n)$, $I_0 = A$

<u>Lemma</u> [AHR2, 6.2] A noetherian and I_1-adically complete.
i.e. (X, X_0) complete.

<u>pf</u>: Fix $N > 0$ ("large"), $\mathcal{X}_0 \overset{\mathcal{I}}{\hookrightarrow} \mathcal{X}_N$, $\mathcal{X}_i \overset{\mathcal{I}^{i+1}}{\hookrightarrow} \mathcal{X}_N$, $\mathcal{I}^{N+1} = 0$

$$\mathrm{Gr}_{\mathcal{I}} \mathcal{O}_{\mathcal{X}_N} = \bigoplus_{i \geq 0}^{N} \mathcal{I}^i / \mathcal{I}^{i+1}$$

"stabilizes when $N \to \infty$" $\Rightarrow$ $\mathrm{Gr}_{\mathcal{I}} \mathcal{O}_{\mathcal{X}} = \bigoplus_{i \geq 0} \mathcal{I}^i / \mathcal{I}^{i+1}$ makes sense even if no $\mathcal{X}_0 \overset{\mathcal{I}}{\hookrightarrow} \mathcal{X}$

an $\mathcal{O}_{\mathcal{X}_0}$-alg of finite type.

$\Rightarrow$ $(\pi_0)_*(\mathrm{Gr}_{\mathcal{I}} \mathcal{O}_{\mathcal{X}})$ finite type $\mathcal{O}_{X_0}$-algs, in particular noeth.

$\quad \| $ coh aff.

$$\bigoplus_{i \geq 0} I_i / I_{i+1} = \mathrm{Gr}_{I_\bullet} A \qquad \text{(NB! non-adic filtration)}$$

[God56, Thm. 4] $\Rightarrow$ A noetherian

(Sém Cartan-Chevalley: Géom Alg. "Topologies m-adiques")
(also Bourbaki)

Let $A \to \hat{A}$ be I_1-adic completion. Then have

$$A \longrightarrow \hat{A} = \varprojlim_n A/(I_1)^n \longrightarrow \varprojlim A/I_n = A$$

identity map. $\Rightarrow$ $\hat{A} \to A$ surjective $\Rightarrow$ A I_1-adic complete. $\square$

(Lemma doesn't use char 0)

<u>Proof of Thm E (char 0) continued</u>

<u>Proof of Thm E</u> when $\mathcal{X}_i / \mathbb{Q}$:

By Lemma have:

$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, hook] \arrow[d] & \mathcal{X}_1 \arrow[r, hook] \arrow[d] & \cdots & \\ X_0 \arrow[r, hook] & X_1 \arrow[r, hook] & \cdots \arrow[r, hook] & X \end{tikzcd}$$

X noetherian

(X, X_0) complete

By assumption (fund) $\mathcal{X}_0 = [\mathrm{aff}/GL_N]$, i.e., $\exists\, \mathcal{X}_0 \xrightarrow{\phi_0} BGL_N$ affine.

<u>Lemma</u> $\exists$ lifts $\phi_n : \mathcal{X}_n \to BGL_N$, ϕ_n affine.

<u>pf</u>: obs to lifting ϕ_n to ϕ_{n+1} lies in $\mathrm{Ext}^1_{\mathcal{X}_n}(L\phi_n^* \underbrace{L_{BGL_N/\mathbb{Q}}}_{\substack{\text{perf of amp. } [1,1] \\ \text{v.b. in deg } 1\ \mathfrak{g}^\vee}}, \mathcal{I}) = 0$

<u>affine</u>: e.g. use Serre's criterion (exc).

<u>Lemma</u> $\exists\, \mathcal{N} \to BGL_N \times X$ affine smooth (a v.b) and closed comp imm $\psi_n : \mathcal{X}_n \hookrightarrow \mathcal{N}$

<u>pf</u>: ~~Have~~ ϕ_n gives affine comp $\mathcal{X}_n \to BGL_n \times X$. Since $\mathcal{X}_n \to X_n$ finite type (GMS (h))

and $BGL_n \times X$ has res prop $\Rightarrow \exists\, \mathcal{X}_1 \overset{\psi_1}{\hookrightarrow} \mathcal{N} \xrightarrow[\text{v.b.}]{} BGL_n \times X$.

Since $\mathcal{N} \to X$ smooth, can lift and $\mathcal{X}_n$ coh aff

$$\begin{tikzcd} \mathcal{X}_{n+1} \arrow[rd, dashed, "\psi_{n+1}"] & \\ \mathcal{X}_n \arrow[u, hook] \arrow[r, "\psi_n"] \arrow[d] & \mathcal{N} \arrow[ld, "\text{smooth}"] \\ X & \end{tikzcd}$$

. ψ_1 closed im $\Rightarrow \psi_n$ closed im $\forall n$ (Lemma 3)

Effectivizes to $\hat{\mathcal{X}} \hookrightarrow \hat{\mathcal{N}} := \mathcal{N} \times_N \hat{N}$ as before (Thm CP).

Proof in general more involved, see [AHR2, §6-10]

4.6 Artin approximation ~~and Artin algebraization~~

S scheme

Artin approximation

$$\begin{matrix} \operatorname{Spec}\widehat{\mathcal{O}}_{S,s} & & \operatorname{Spec}\mathcal{O}^h_{S,s} & & & & \\ \| & & \| & & & & \\ S^\wedge_s & \longrightarrow & S^h_s & \longrightarrow & S' & \xrightarrow[\substack{\text{étale}\\ \text{nbhd}\\ \text{of } s}]{} & S \end{matrix}$$

Given an "object" over $S^\wedge_s$, does ~~it~~ there exist approximate object over S^h_s?

over some étale nbhd S'?

Ex: object = vector bundle, coherent sheaf, family of curves, finite étale covers, ...

Assume S excellent.

Thm AA: Let $F: (Sch/S)^{op} \to Set$ be a functor, loc. of fin. pres.

(i.e. if $B = \varinjlim A_\lambda$, then $F(\operatorname{Spec} B) = \varinjlim F(\operatorname{Spec} A_\lambda)$).

Given $\bar{\xi} \in F(S^\wedge_s)$ and $N \geq 0$, $\exists \xi \in F(S^h_s)$ (or $\xi \in F(S')$ for some $S' \to S$ étale nbhd)

s.th. $\bar{\xi}$ and ξ equal in $F(S^{[N]}_s)$.

Rmk: Also true for pairs. (S affine, repl s with closed subscheme S_0)

$$(S^\wedge_{S_0}, S_0) \to (S^h_{S_0}, S_0) \to (S', S_0) \xrightarrow{\text{étale}} \underset{\text{affine}}{(S, S_0)}$$

pf: Néron–Popescu desingularization + small arg in local case (S,s)

more serious arg in case (S, S_0) Elkik.

$$S^\wedge = \varprojlim_{\substack{S' \to S\\ \text{smooth}}} S'$$

(Rmk: Sometimes $\bar{\xi} \approx \hat{\xi} \Rightarrow \bar{\xi} = \hat{\xi}$. Ex finite étale covers

($\hat{\xi} \in \Gamma(\hat{S})$ image of ξ))

4.7 Proof of Step 4 (smooth case)

Step 1-3 gives $\hat{\mathcal{W}} \xrightarrow{\hat f} \mathcal{X}$ extending f_0

Recall $\mathcal{N} = [\text{normal space}/G_x] = \mathrm{Spec}_{BG_x} \mathrm{Sym}(\mathcal{I}/\mathcal{I}^2)$ $\qquad \mathcal{W}_0 \overset{\mathcal{I}}{\hookrightarrow} \hat{\mathcal{W}}$

By construction $\hat{\mathcal{W}} \overset{\hat\phi}{\hookrightarrow} \hat{\mathcal{N}}$ s.th ϕ_1 iso. But $\mathcal{X}$ smooth $\Rightarrow$ $\mathrm{Gr}_{\mathcal{I}}\, \mathcal{O}_{\hat{\mathcal{W}}} = \mathrm{Sym}\, \mathcal{I}/\mathcal{I}^2$

$\Rightarrow$ $\hat\phi$ iso.

$$
\begin{tikzcd}
\hat{\mathcal{W}} \arrow[d, equal] \arrow[rrr, bend left, "\hat f"] \arrow[dr, phantom, "(*)"] & & & \mathcal{X} \\
\hat{\mathcal{N}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{N} \times_N T \arrow[r] \arrow[d] \arrow[urr, dashed, "f"] \arrow[dr, phantom, "\square"] & \mathcal{N} \arrow[d] \\
\hat{N} \arrow[r] & T \arrow[r, "\text{ét}"] & N
\end{tikzcd}
$$

$*$ does not commute, only up to 1st order

Artin approximation for

$$F : (\mathrm{Sch}_{/N})^{op} \longrightarrow (\mathrm{Set})$$
$$(T \to N) \longmapsto \pi_0 \mathrm{Map}(\mathcal{N} \times_N T, \mathcal{X})$$

gives $T \to N$ étale nbhd and $f : \mathcal{N} \times_N T \to \mathcal{X}$ s.th $f_1 = \hat f_1$

Lemma: $\hat f$ formally smooth + $\hat{\mathcal{N}}$ complete, $f_1 = \hat f_1$ $\Rightarrow$ ∃ iso $\alpha : \hat{\mathcal{W}} \to \hat{\mathcal{N}}$ s.th. (*) commutes after replacing $\hat f$ with $\hat f \circ \alpha$.

$\Rightarrow$ f smooth.

Better diagram:

$$
\begin{tikzcd}
 & & \hat{\mathcal{N}} \arrow[dl] \arrow[dd] \arrow[dr, "\hat f"] \arrow[d, phantom, "(*)"] & \\
 & \mathcal{W} \arrow[dl, "\text{ét}"', "g"] \arrow[dd] \arrow[rr, "f"] \arrow[dr, phantom, "\square"] & & \mathcal{X} \\
\mathcal{N} \arrow[dd] \arrow[dr, phantom, "\square"] & & \hat{N} \arrow[dl] & \\
 & T \arrow[dl, "\text{ét}"] & & \\
N & & &
\end{tikzcd}
$$

4.8 Artin algebraization

Question: Given complete local ring $\bar{A}$, when is $\bar{A} = \hat{A}$ for a fin. generated algebra A ?

Ex: $\bar{A}$ regular $\Rightarrow$ $\underset{\bar{A}}{\underset{\shortparallel}{k[[x_1, \dots, x_n]]}}$ $\Rightarrow$ can take $A = k[x_1, \dots, x_n]$

Stacky analogue: $\hat{\mathcal{W}} = \hat{\mathcal{N}}$ in the smooth case.

In general, answer is no but positive answer in following case:

Thm (Artin alg.) Given (alg) stack $\mathcal{X}$, loc. of fin. pres / excellent base S and $\operatorname{Spec} \bar{A} \xrightarrow{\bar{m}} \mathcal{X}$ formally smooth, then $\exists A$ s.th $\bar{A} = \hat{A}$ and

$$\begin{tikzcd} \operatorname{Spec} \bar{A} \arrow[rr, bend left, "\bar{m}"] \arrow[dr, "\text{completion}"'] & \circ & \mathcal{X} \\ & \operatorname{Spec} A \arrow[ur, "m"'] \arrow[dr, "\text{f.t.}"] & \\ & & S \end{tikzcd}$$

In part m also (form.) smooth.

Thm EAA (equi. Artin Alg) over a field k [AHR1, A.19]

$\mathcal{X}$ alg stack (w/ aff. stab) f.t./k. $\hat{\mathcal{H}} = [\operatorname{Spec} C/H] \xrightarrow{\hat{f}} \mathcal{X}$ form. smooth, H lin red.

Then $\exists\, \mathcal{W} = [\operatorname{Spec} A/H]$ s.th $\hat{\mathcal{H}} = \hat{\mathcal{W}}$ and:

$$\begin{tikzcd} \hat{\mathcal{H}} \arrow[rr, bend left, "\hat{f}"] \arrow[dr, "\text{completion}"'] & \circ & \mathcal{X} \\ & \mathcal{W} \arrow[ur, "f"'] \arrow[dr, "\text{f.t.}"] & \\ & & \operatorname{Spec} k \end{tikzcd}$$

In part, f smooth.

pf: ~~Use~~ Write $\hat{\mathcal{H}} \hookrightarrow \hat{N}$. $E \xrightarrow{\alpha} \mathcal{O}_{\hat{N}}^{\oplus r} \xrightarrow{\beta} \mathcal{O}_{\hat{N}} \longrightarrow \mathcal{O}_{\hat{\mathcal{H}}} \to 0$.

Use A.A and refined Rees lemma to get

$F \xrightarrow{\tilde{\alpha}} \mathcal{O}_{W \times_N T}^{\oplus r} \xrightarrow{\tilde{\beta}} \mathcal{O}_{W \times_N T} \longrightarrow \mathcal{O}_{\mathcal{W}} \to 0$ s.th $\mathcal{W}$ and $\hat{\mathcal{H}}$ have iso tangent cones.

Use $\hat{f}$ form. versal to obtain $\alpha : \hat{\mathcal{H}} \xrightarrow{\sim} \hat{\mathcal{H}}$ s.th. (*) commutes.

4.9 <u>Proof of Step 4</u> (general case)

Step 1 – 3 gives
$$\begin{tikzcd} \hat{\omega} \arrow[r, "\hat{f}"] \arrow[d, equal] & \mathcal{H} \\ {[\mathrm{Spec}\, C/H]} & \end{tikzcd}$$
form smooth

Apply EAA with $\hat{\mathcal{M}} = \hat{\omega}$.