

LECTURE 4: Tannaka duality, Artin approximation and equi. Artin algebraization

- 4.1 Tannaka duality (Thm TD)
- 4.2 Effectivity (Thm E)
- 4.4 Proof of effectivity over a field
- ↪ 4.3 Proofs of Step 2 and 3 of local str. thm
- 4.5 Proof of effectivity in general / char 0.
- 4.6 Artin approximation (Thm AA)
- 4.7 Proof of Step 4 (smooth case)
- 4.8 (Equivalent) Artin algebraization (Thm EAA)
- 4.9 Proof of Step 4 (general case)

4.1 Tannaka duality

$f: \mathcal{X} \rightarrow \mathcal{Y}$ induces $f^*: \text{QCoh}(\mathcal{Y}) \rightarrow \text{QCoh}(\mathcal{X})$
 (also $f^*: \text{Coh}(\mathcal{Y}) \rightarrow \text{Coh}(\mathcal{X})$, if \mathcal{X}, \mathcal{Y} noeth)

s.th. (o) f^* additive

(i) $f^* \mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{X}}$

(ii) $f^*(F \otimes_{\mathcal{O}_{\mathcal{Y}}} G) = f^* F \otimes_{\mathcal{O}_{\mathcal{X}}} f^* G$

(iii) f^* cocontinuous (preserves colimits), i.e.;

right exact and $f^*(\bigoplus F_{\alpha}) = \bigoplus (f^* F_{\alpha})$

} (symm.) monoidal functor

(f^* right-exact for Coh version)

Theorem TD:
 [HR19, Thm 1.1]

\mathcal{X} excellent alg stack
 \mathcal{Y} noetherian alg stack w/ affine stabilizers
 (no other sep. assump!)

Variant:
 \mathcal{X} noetherian
 \mathcal{Y} noetherian, $\Delta_{\mathcal{Y}}$ qaff

Then:

$$\text{Map}(\mathcal{X}, \mathcal{Y}) \longrightarrow \text{Hom}_{\text{r}\otimes, \simeq}(\text{Coh}(\mathcal{Y}), \text{Coh}(\mathcal{X}))$$

$$f \longmapsto f^*$$

equivalence of groupoids.

objects are right-exact tensor functor (additive + monoidal)
 \simeq = morphisms are natural isomorphisms

Corollary: If $(\mathcal{X}, \mathcal{X}_0)$ excellent complete pair w/ aff stab.

then $\mathcal{X} = \varinjlim_n \mathcal{X}^{[n]}$ in category of noeth alg stacks w/ aff stab.

proof:

$$\begin{aligned} \text{Map}(\mathcal{X}, \mathcal{Y}) &\stackrel{\text{TD}}{=} \text{Hom}_{\text{r}\otimes, \simeq}(\text{Coh}(\mathcal{Y}), \text{Coh}(\mathcal{X})) \\ &\stackrel{\text{CP}}{=} \text{Hom}_{\text{r}\otimes, \simeq}(\text{Coh}(\mathcal{Y}), \varprojlim \text{Coh}(\mathcal{X}^{[i]})) \\ &= \varprojlim \text{Hom}_{\text{r}\otimes, \simeq}(\text{Coh}(\mathcal{Y}), \text{Coh}(\mathcal{X}^{[i]})) \\ &\stackrel{\text{TD}}{=} \varprojlim \text{Map}_{\text{r}\otimes, \simeq}(\mathcal{X}^{[i]}, \mathcal{Y}) \end{aligned}$$

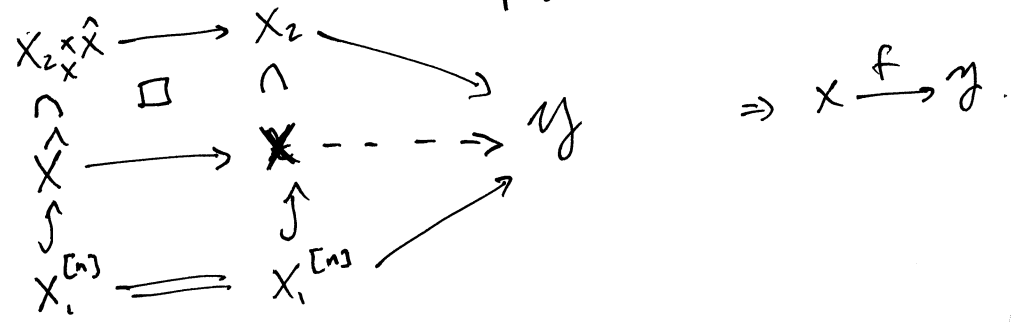
Remarks: • TD \Rightarrow AlgStacks_{exc, aff stab} $\xrightarrow{\text{fully faithful}}$ ATC = Grothendieck abelian tensor categories

$\mathcal{X} \xrightarrow{\quad} \text{QCoh}(\mathcal{X})$

- aff stab crucial: $\text{QCoh}(BA) = \text{QCoh}(*)$ if $A \rightarrow *$ abelian variety.
 - \otimes crucial: $\text{QCoh}(B\mu_2) \cong \text{QCoh}(*) \amalg *$ as abelian categories
- w/ \otimes : LHS = $(\mathbb{Z}/2\mathbb{Z})$ -graded vsp $V_0 \oplus V_1$
 RHS = (V, W) V, W v.sp.
- Derived version w/ D_{qc} instead of QCoh (Bhatt-Halpern-Leistner) for quasi-affine diagonal.

Outline of proof:

- (1) \mathcal{Y} aff stab $\Leftrightarrow \exists$ stratification w/ strata $\mathcal{Y}_i = [\text{qaff}/G_i]$
 - (2) TD holds for $\mathcal{Y} = \mathcal{Y}_i$ (easy: use that vector bundles are preserved by tensor functors b/c v.b = dualizable)
 - (3) Main lemma: TD holds for $\mathcal{Y} = \mathcal{Y}_i^{[n]}$ $\forall n$.
 - (4) Induction on # strata \Rightarrow WLOG $\mathcal{Y}_1 \xrightarrow{\text{closed}} \mathcal{Y} \supseteq \mathcal{Y}_2 = \mathcal{Y}_1 \amalg \mathcal{Y}_1$
- "pullback" to \mathcal{X} : $\mathcal{X}_1 \xrightarrow{\quad} \mathcal{X} \supseteq \mathcal{X}_2$.
- (5) Tensor localization gives tensor functor $\text{Coh}(\mathcal{Y}_2) \rightarrow \text{Coh}(\mathcal{X}_2) \xrightarrow{\text{induction}} \mathcal{X}_2 \rightarrow \mathcal{Y}_2$
- Step (3) gives $\mathcal{X}_1^{[n]} \rightarrow \mathcal{Y}_1^{[n]}$
- (6) Formal gluings "Mayer-Vietoris" gives (if \mathcal{X} excellent) (wlog $\mathcal{X} = X$ affine by descent)



4.2 Effectivity

Def: An adic sequence of stacks is a diagram of closed immersions

[AHR2, 1.8]

$$\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \mathcal{X}_2 \hookrightarrow \dots$$

s.t. $\forall i < j, \mathcal{X}_i = i^{\text{th}}$ inh nbhd of \mathcal{X}_0 in \mathcal{X}_j .

Ex: If $\mathcal{X}_0 \hookrightarrow \mathcal{X}$, then $\mathcal{X}_*^{[0]} \hookrightarrow \mathcal{X}^{[1]} \hookrightarrow \dots$ adic seq.

Def: A coherent completion of an adic seq $\{\mathcal{X}_n\}$ is a stack $\hat{\mathcal{X}}$ s.t.

[AHR2, 1.9]

- (1) $\exists \mathcal{X}_i \hookrightarrow \hat{\mathcal{X}}$ comp. closed immersions.
- (2) $\hat{\mathcal{X}}$ noeth, w/ affine diag
- (3) $(\hat{\mathcal{X}}, \mathcal{X}_0)$ complete.

Rmk: TD $\Rightarrow \hat{\mathcal{X}} = \varinjlim \mathcal{X}_i$ in category of noeth stacks w/ qaff diag (or exc stacks w/ aff stab)

Ex: $\hat{\mathcal{X}}_x := \text{coh. completion of } \{\mathcal{X}_x^{[n]}\}$.

Thm E: $\{\mathcal{X}_i\}$ adic seq of noeth stacks.

[AHR2, 1.10] \mathcal{X}_0 lin fund $\Rightarrow \hat{\mathcal{X}}$ exists and is lin fund.

Rmk: Have seen that \mathcal{X}_i lin fund. Thus here:

$$\begin{array}{ccccccc} \mathcal{X}_0 & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & & & \\ \downarrow & & \downarrow & & & & \\ \mathcal{X}_0 & \hookrightarrow & \mathcal{X}_1 & \hookrightarrow & \dots & & \\ \text{"} & & \text{"} & & & & \\ \text{Spec } A_0 & & \text{Spec } A_1 & & & & \end{array}$$

Problem: $\mathcal{X}_0 \hookrightarrow \mathcal{X}_1 \hookrightarrow \dots$ not adic.

We will have that $\hat{\mathcal{X}} = \varinjlim \mathcal{X}_n$ gms of $\hat{\mathcal{X}}$ but a priori not even noetherian!

49 Proofs of Steps 2 and 3

$$\begin{array}{ccc} \omega_0 & \xrightarrow{f_0} & \mathbb{X}_0 \quad \text{étale/smooth/synt.} \\ \text{lin fund} & & \downarrow \\ & & \mathbb{X} \end{array}$$

Step 1 gives adic seq $\{\omega_i\}$ and $\omega_i \xrightarrow{f_i} \mathbb{X}^{[i]}$

Step 2: Thm E $\Rightarrow \exists \hat{\omega} = \varinjlim \omega_i$ lin fund, $(\hat{\omega}, \omega_0)$ coh. complete.

Step 3: Thm TD $\Rightarrow \exists \hat{f}: \hat{\omega} \rightarrow \mathbb{X}$ extending the f_i .

\hat{f} flat (loc crit of flatness) but not necc of finite type

4.4 Proof of effectivity when $\mathcal{X}/\text{Spec } k$ and $X_0 = \text{Spec } k$, $h = \bar{h}$. [AHR1, pt Thm 1.1]

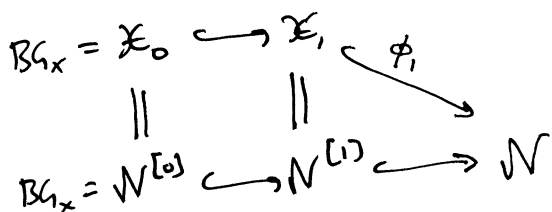
Let $\frac{\mathcal{X}_1}{\mathcal{X}_0} = \text{conormal sheaf of } \mathcal{X}_0 \hookrightarrow \mathcal{X}_1$.

$\frac{\mathcal{X}_1}{\mathcal{X}_0} \in \text{Coh}(\mathcal{X}_0) = \text{VB}(\text{BG}_X) = \text{bindim } G_X\text{-rep. } V$

Let $N = \text{Spec } \text{Sym}_{\text{BG}_X}(\mathcal{X}/\mathcal{I}^2) = [V/G_X]$ normal stack. (smooth)

Lemma 1: All square-zero extensions of \mathcal{X}_0 w/ ideal \mathcal{J} are isomorphic (all trivial)

pf: $\text{Ext}_k^1(\text{BG}_X, \mathcal{J}) = \text{Ext}_{\text{BG}_X}^1(L_{\text{BG}_X/k}, \mathcal{J}) = H^1(\text{BG}_X, \underbrace{L_{\text{BG}_X/k}^\vee \otimes \mathcal{J}}_{\text{Tor-amp } [-1,0]}) = 0$
 b/c $\text{BG}_X \rightarrow \text{Spec } k$ smooth



Lemma 2: We can lift $\mathcal{X}_n \xrightarrow{\phi_n} N$ to $\mathcal{X}_{n+1} \xrightarrow{\phi_{n+1}} N \quad \forall n$.

pf: obs $\in \text{Ext}_{\mathcal{X}_n}^1(\phi_n^* L_{N/k}, \mathcal{J}) = H^1(\mathcal{X}_n, \underbrace{\phi_n^* L_{N/k}^\vee \otimes \mathcal{J}}_{\text{Tor-amp } [-1,0]}) = 0$
 b/c $N \rightarrow \text{Spec } k$ smooth.

Lemma 3: ϕ_1 closed immersion $\Rightarrow \phi_n$ closed imm $\forall n$.

[AHR1, Prop A.8 (1)]

[AHR2, Lem 6.3]

Follows from:

Lemma 4: $g: A \rightarrow B$ s.t. $A \rightarrow B \rightarrow B/I$ surj and $I^n = 0$. Then g surjective.

pf: Use Nakayama (for nilpotent ideals) twice: first on B to show $\ker(g)B = I$.

Then on A to show that $A \rightarrow B$ surj.

proof of effectivity over field \bar{k} continued

Since $\{X_i\}$ adic and $\phi_n: X_n \hookrightarrow \mathcal{N}$ closed imm, we get:

$$\begin{array}{ccccccc} X_0 & \hookrightarrow & X_1 & \hookrightarrow & X_2 & \hookrightarrow & X_3 & \hookrightarrow & \dots \\ \parallel & \square & \parallel & \square & \downarrow \text{closed} & \square & \downarrow \text{closed} & & \\ N_0^{[0]} & \hookrightarrow & N_1^{[1]} & \hookrightarrow & N_2^{[2]} & \hookrightarrow & N_3^{[3]} & \hookrightarrow & \dots \end{array}$$

$\Rightarrow \{O_{X_n}\} \in \varprojlim \text{Coh}(N^{[n]})$

Def: $N_{\text{gms}}(T(N, O_N))$ i.e. gms of N .

Def: $\hat{N} := N \times_N N^\wedge$

$$\begin{array}{ccc} \hat{N} & \longrightarrow & N \\ \text{gms} \downarrow & \square & \downarrow \text{gms} \\ N^\wedge & \longrightarrow & N \end{array}$$

Thm CP $\Rightarrow (\hat{N}, 0)$ complete, i.e. $\text{Coh}(\hat{N}) = \varprojlim \text{Coh}(N^{[n]})$

$\Rightarrow O_{\hat{X}} \in \text{Coh}(\hat{N})$ effectivizing $\{O_{X_n}\}$.

This is an $O_{\hat{N}}$ -alg equipped w/ $O_{\hat{N}} \rightarrow O_{\hat{X}}$, i.e.

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\phi} & \hat{N} \text{ closed imm. s.th.} \\ X_n & \hookrightarrow & \hat{X} \\ \downarrow & \square & \downarrow \\ N^{[n]} & \hookrightarrow & \hat{N} \end{array} \quad \forall n.$$

• Verify (Exc) that (\hat{X}, X_0) coh. complete

• By construction $\hat{X} \hookrightarrow \hat{N} \rightarrow N \rightarrow \text{BG}_x$ affine, i.e., linearly fundamental □

4.5 Proof of Thm E (char 0)

[AHRZ, §6]

$X_0 \hookrightarrow X_1 \hookrightarrow \dots$ adic seq of (lin) fund stacks / \mathbb{Q} ^{noetherian}
 $\pi_0 \downarrow \quad \downarrow$
 $X_0 \hookrightarrow X_1 \hookrightarrow$ their gms, not adic.

$$X_n = \text{Spec } A_n$$

$$X = \text{Spec } A, \quad A = \varprojlim A_n$$

Lemma A noetherian and I_1 -adically complete.

$$I_{n+1} = \ker(A \rightarrow A_n), \quad I_0 = A$$

[AHRZ, 6.2] i.e. (X, X_0) complete.

pf: Fix $N > 0$ ("large"), $X_0 \xrightarrow{I} X_N, \quad X_i \xrightarrow{I^{i+1}} X_N, \quad I^{N+1} = 0$

$$\text{Gr}_I^0 \mathcal{O}_{X_N} = \bigoplus_{i \geq 0}^N I^i / I^{i+1}$$

"stabilizes when $N \rightarrow \infty$ " $\Rightarrow \text{Gr}_I^0 \mathcal{O}_X = \bigoplus_{i \geq 0} I^i / I^{i+1}$ makes sense even if no $X_0 \xrightarrow{I} X$

an \mathcal{O}_{X_0} -alg of finite type.

$\Rightarrow (\pi_0)_* (\text{Gr}_I^0 \mathcal{O}_X)$ finite type \mathcal{O}_{X_0} -alg, in particular noeth.

|| coh. aff.

$$\bigoplus_{i \geq 0} I_i / I_{i+1} = \text{Gr}_{I_0} A \quad (\text{NB! non-adic filtration})$$

[God56, Thm. 4] $\Rightarrow A$ noetherian

(Sém Cartan-Chevalley: Géom Alg. "Topologies m-adiques")

(also Bourbaki)

Let $A \rightarrow \hat{A}$ be I_1 -adic completion. Then have

$$A \rightarrow \hat{A} = \varprojlim_n A / (I_1)^n \rightarrow \varprojlim_n A / I_n = A$$

identity map. $\Rightarrow \hat{A} \rightarrow A$ surjective $\Rightarrow A$ I_1 -adic complete. \square

(Lemma doesn't use char 0)

4.6 Artin approximation and Artin algebraization

S scheme

Artin approximation

$$\begin{array}{ccccccc}
 \text{Spec } \hat{\mathcal{O}}_{S,s} & & \text{Spec } \mathcal{O}_{S,s}^h & & & & \\
 \Downarrow & & \Downarrow & & & & \\
 S^\wedge & \longrightarrow & S_s^h & \longrightarrow & S' & \longrightarrow & S \\
 & & & & \text{\small \textit{étale} \\ \text{\small \textit{nbhd} \\ \text{\small \textit{of } S}} & &
 \end{array}$$

Given an "object" over S^\wedge , does ~~it~~ ^{there exist} approximate object over S_s^h ?
 over some étale nbhd S' ?

Ex: object = vector bundle, coherent sheaf, family of curves, finite étale covers, ...

Assume S excellent.

Thm AA: Let $F: (\text{Sch}/S)^{\text{op}} \rightarrow \text{Set}$ be a functor, loc. of fin. pres.

(i.e. if $B = \varinjlim A_i$, then $F(\text{Spec } B) = \varinjlim F(\text{Spec } A_i)$).

Given $\bar{\xi} \in F(S^\wedge)$ and $N \geq 0$, $\exists \xi \in F(S_s^h)$ (or $\xi \in F(S')$ for some $S' \rightarrow S$ étale nbhd)
 s.t. $\bar{\xi}$ and ξ equal in $F(S_s^{[N]})$.

Remk: Also true for pairs. (S affine, repl s with closed subscheme S_0)

$$(S_{S_0}^\wedge, S_0) \longrightarrow (S_{S_0}^h, S_0) \longrightarrow (S', S_0) \xrightarrow{\text{\small \textit{étale}}} (S, S_0)$$

affine

pf: Néron-Popescu desingularization

+ small org in local case (S, s)

more serious org in case (S, S_0) \in l.h.

$$S^\wedge = \varprojlim_{\substack{S' \rightarrow S \\ \text{smooth}}} S'$$

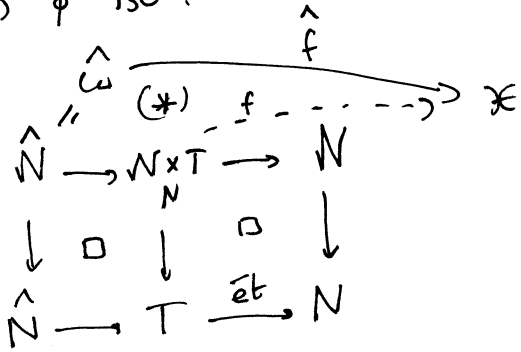
Remk: Sometimes $\bar{\xi} \approx \hat{\xi} \Rightarrow \bar{\xi} = \hat{\xi}$. Ex finite étale covers
 ($\hat{\xi} \in \Gamma(\hat{S})$ image of $\bar{\xi}$)

4.7 Proof of Step 4 (smooth case)

Step 1-3 gives $\hat{\omega} \xrightarrow{\hat{f}} \mathcal{X}$ extending f_0

Recall $N = [\text{normal space}/G_x] = \text{Spec}_{\mathbb{R}x} \text{Sym}(\mathcal{I}/\mathcal{I}^2)$ $\omega_0 \xrightarrow{\mathcal{I}} \hat{\omega}$

By construction $\hat{\omega} \xrightarrow{\hat{\phi}} \hat{N}$ s.t. ϕ_1 iso. But \mathcal{X} smooth $\Rightarrow \text{Gr}_{\mathcal{I}} \hat{\omega} = \text{Sym } \mathcal{I}/\mathcal{I}^2$
 $\Rightarrow \hat{\phi}$ iso.



* does not commute, only up to 1st order

Artin approximation for

$$F: (\text{Sch}/N)^{\text{op}} \rightarrow (\text{Set})$$

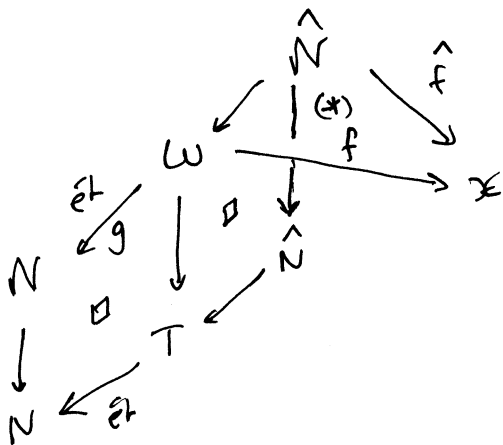
$$(T \rightarrow N) \mapsto \pi_0 \text{Map}(N \times_T N, \mathcal{X})$$

gives $T \rightarrow N$ étale nbhd and $f: N \times_T N \rightarrow \mathcal{X}$ s.t. $f_1 = \hat{f}_1$

Lemma: \hat{f} formally smooth + \hat{N} complete \Rightarrow ~~also~~ \exists iso $\alpha: \hat{N} \rightarrow \hat{N}$ s.t. $(*)$ commutes after replacing \hat{f} with $\hat{f} \circ \alpha$.

$\Rightarrow f$ smooth.

Better diagram:



4.8 Artin algebraization

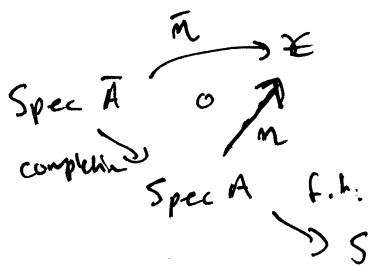
Question: Given complete local ring \bar{A} , when is $\bar{A} = \hat{A}$ for a fin. generated algebra A ?

Ex: \bar{A} regular $\Rightarrow \hat{\bar{A}} = k[[x_1, \dots, x_n]] \Rightarrow$ can take $A = k[x_1, \dots, x_n]$

Stacky analogue: $\hat{W} = \hat{N}$ in the smooth case.

In general, answer is no but positive answer in following case:

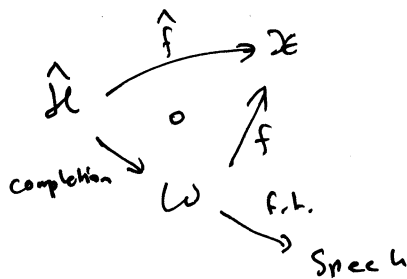
Thm (Artin alg.) Given (alg) stack \mathcal{X} , loc. of fin. pres / excellent base S and $\text{Spec } \bar{A} \xrightarrow{\bar{m}} \mathcal{X}$ formally smooth, then $\exists A$ s.t. $\bar{A} = \hat{A}$ and



In part m also (form.) smooth.

Thm EAA (equiv. Artin Alg) \mathcal{X} alg stack (w/ aff. stab) f.t./k. over a field k . $\hat{\mathcal{H}} = [\text{Spec } C/H] \xrightarrow{\hat{f}} \mathcal{X}$ form. smooth, H lin. red. [AHRI, A.19]

Then $\exists W = [\text{Spec } A/H]$ s.t. $\hat{\mathcal{H}} = \hat{W}$ and:



In part, f smooth.

pf: Use $\hat{\mathcal{H}} \subset \hat{N}$. $E \xrightarrow{\alpha} \mathcal{O}_{\hat{N}} \xrightarrow{\beta} \mathcal{O}_{\hat{\mathcal{H}}} \rightarrow 0$.

Use A.A and refined Rees lemma to get

$$F \xrightarrow{\tilde{\alpha}} \mathcal{O}_{W \times T} \xrightarrow{\tilde{\beta}} \mathcal{O}_{W \times T} \rightarrow \mathcal{O}_W \rightarrow 0 \quad \text{s.t. } W \text{ and } \hat{\mathcal{H}} \text{ have iso tangent cones.}$$

Use \hat{f} form. versal to obtain $\alpha: \hat{\mathcal{H}} \xrightarrow{\sim} \hat{\mathcal{H}}$ s.t. $(*)$ commutes.

4.9 Proof of Step 4 (general case)

Step 1-3 gives $\hat{\omega} \xrightarrow{\hat{f}} X$ form smooth
" $[\text{Spec } C/H]$

Apply EAA with $\hat{\mathcal{M}} = \hat{\omega}$.