LECTURE 3 Deformation theory and coherent completeness

3.1 Cotangent complexes

3.2 Deformation theory

3.3 Formal neighborhoods (pf of Step 1)

3.4 Complete and henselian pairs (Thm HP, Thm CP)

3.5 Applications of Thm HP : Luna's fund. lemma and univ. of gms

3.6 Examples of complete stacks

3.7 Formal functions (Thm FF)

3.8 Coherent completeness w/ res. prop (pf of Thm CP)

3.1 Deformation theory: Cotangent complex

Cotangent complex

$\mathcal{X} \xrightarrow{f} \mathcal{Y} \quad \rightsquigarrow \quad \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in D_{qc}(\mathcal{X})$

Properties

- f repr. (or rel DM) $\Rightarrow$ $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in D^{(-\infty,0]}$
- f rel n-stack $\Rightarrow$ $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in D^{(-\infty,n]}$ (we only care about $n=1$)
- f smooth $\Rightarrow$ $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ perfect of Tor-amp $[0,1]$ ← 1-stack
- f smooth gerbe $\Rightarrow$ —— " —— $[1,1]$
- f étale $\Rightarrow$ $\mathbb{L}_{\mathcal{X}/\mathcal{Y}} = 0$
- f syntomic $\Rightarrow$ —— " —— $[-1,1]$

Fund exact Δ

$$\mathcal{X} \xrightarrow{f} \mathcal{Y} \xrightarrow{g} \mathcal{Z} \quad \Rightarrow \quad f^*\mathbb{L}_{\mathcal{Y}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Z}} \to \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \xrightarrow{+1}$$

Exc 3.1 (a) G smooth$/k$. $\mathbb{L}_{BG/k} = \mathfrak{g}^\vee[-1]$

Exc (b) $\mathcal{X} = [U/G]$ $\qquad p^*\mathbb{L}_{\mathcal{X}/k} = [\underset{0}{\Omega_{U/k}} \to \underset{1}{\mathfrak{g}^\vee}]$

U smooth$/k$, G smooth$/k$

$p : U \to \mathcal{X}$

3.2 Deformation theory (of stacks)

Problem 1

[01506, Thm 1.1]

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "I"] \arrow[d, "f"'] & \mathcal{X}' \arrow[dl, "f'"] \arrow[d, phantom, "\circ"] \\ \mathcal{Y} & \end{tikzcd}$$ sq zero ext

(f representable)
but not necc

$$\mathrm{Exal}_{\mathcal{Y}}(\mathcal{X}, I) = \mathrm{Ext}^1(L_{\mathcal{X}/\mathcal{Y}}, I)$$

$\shortparallel$

set of iso-classes of extensions

Problem 2

[01506, Thm 1.4]

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d, "f", "\text{flat}"'] \arrow[dr, phantom, "\square"] & \mathcal{X}' \arrow[d, "f", "\text{flat}"'] \\ \mathcal{Y} \arrow[r, hook, "j", "J"'] & \mathcal{Y}' \end{tikzcd}$$ sq zero

(f representable)

(i) $\exists\, o(f,j) \in \mathrm{Ext}^2(L_{\mathcal{X}/\mathcal{Y}}, f^*J)$

obstruction to $\exists\, \mathcal{X}'$.

(ii) if $o(f,j)=0$, then set of iso-classes of $\mathcal{X}'$ is a torsor under

$\mathrm{Ext}^1(L_{\mathcal{X}/\mathcal{Y}}, f^*J)$

(iii) $\mathrm{Aut}(\mathcal{X}'/\mathcal{X}, \mathcal{Y}') = \mathrm{Ext}^0(L_{\mathcal{X}/\mathcal{Y}}, f^*J)$

Problem 3

[01506, Thm 1.5]

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i", "I"'] \arrow[d, "f"] \arrow[dd, bend right, "g"'] & \mathcal{X}' \arrow[d, dashed, "f'"] \arrow[dd, bend left, "g'"] \\ \mathcal{Y} \arrow[r, hook, "j", "J"'] \arrow[d, "h"] & \mathcal{Y}' \arrow[d, "h'"] \\ \mathcal{Z} \arrow[r, hook, "k", "K"'] & \mathcal{Z}' \end{tikzcd}$$

ε specified $\begin{tikzcd} g^*K \arrow[r] \arrow[rr, bend right] & f^*J \arrow[r, "\varepsilon"] & I \end{tikzcd}$

We'll either have
(a) $\mathcal{Z} = \mathcal{Z}'$, $\mathcal{Y} = \mathcal{Y}'$, $K = J = 0$, $\varepsilon = 0$
(b) cartesian squares, ε iso.

(i) $\exists\, o(f,i,j,\varepsilon) \in \mathrm{Ext}^1(f^*L_{\mathcal{Y}/\mathcal{Z}}, I)$

(ii) if $o=0$, set of lifts f' up to iso is torsor under $\mathrm{Ext}^0(f^*L_{\mathcal{Y}/\mathcal{Z}}, I)$

(iii) $\mathrm{Aut}(f'/f) = \mathrm{Ext}^{-1}(f^*L_{\mathcal{Y}/\mathcal{Z}}, I)$

3.3 Proof of Step 1: formal neighborhoods

Setup: [AHR 1]

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "f_0"] \arrow[d, equal] & \mathcal{X}_0 \arrow[d, equal] \\ BH & BG_x \end{tikzcd}$$

Étale/smooth

[AHR 2]

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "f_0"] \arrow[d, equal] & \mathcal{X}_0 \arrow[d, equal] \\ \mathcal{H} & \mathcal{G}_x \end{tikzcd}$$

$\mathcal{H}$: lin fund one-pt gerbe

— " —

[AHR 3]

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "f_0"] & \mathcal{X}_0 \arrow[d, hook, "\text{closed substack}"] \\ & \mathcal{X} \end{tikzcd}$$

$\mathcal{W}_0$ lin. fund

Étale/smooth/(syntomic)

Def: $\mathcal{X}^{[n]}$ = nth inf nbhd of $\mathcal{X}_0$.

$\mathcal{X}_0 = V(\mathcal{I}) \hookrightarrow \mathcal{X}$

$\mathcal{X}^{[n]} = V(\mathcal{I}^{n+1}) \hookrightarrow \mathcal{X}$

Lemma: $\exists$ alg stacks $\mathcal{W}_i$ and étale/sm/synt over $\mathcal{X}^{[i]}$

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, hook] \arrow[d, "f_0"'] \arrow[dr, phantom, "\square"] & \mathcal{W}_1 \arrow[r, hook] \arrow[d, "f_1"'] \arrow[dr, phantom, "\square"] & \mathcal{W}_2 \arrow[r, hook] \arrow[d, "f_2"'] \arrow[dr, phantom, "\square"] & \cdots \\ \mathcal{X}^{[0]} \arrow[r, hook] & \mathcal{X}^{[1]} \arrow[r, hook] & \mathcal{X}^{[2]} \arrow[r, hook] & \cdots \end{tikzcd}$$

pf: $\mathcal{X}^{[i]} \hookrightarrow \mathcal{X}^{[i+1]}$ sqzero extension by $\mathcal{I}^{i+1}/\mathcal{I}^{i+2} =: \mathcal{J}$

If we have f_i then $o(f_{i+1}) \in \mathrm{Ext}^2_{\mathcal{W}_i}(\mathbb{L}_{\mathcal{W}_i/\mathcal{X}^{[i]}}, \cancel{\mathcal{I}^{i+1}/\mathcal{I}^{i+2}}\ f_i^*\mathcal{J})$

(Deformation problem 2)

$= \mathrm{Ext}^2_{\mathcal{W}_0}(\mathbb{L}_{\mathcal{W}_0/\mathcal{X}^{[0]}}, \cancel{\mathcal{I}^{i+1}/\mathcal{I}^{i+2}}\ f_i^*\mathcal{J})$

$= H^2(\mathcal{W}_0, \mathbb{L}^\vee_{\mathcal{W}_0/\mathcal{X}^{[0]}} \otimes \cancel{\mathcal{I}^{i+1}/\mathcal{I}^{i+2}}\ f_i^*\mathcal{J}) = 0$

b/c $\mathcal{W}_0$ coh. affine and $\mathbb{L}^\vee_{f_0}$ perfect of Tor-amplitude
$\emptyset$ if f_0 étale
$[-1, 0]$ if f_0 smooth
$[-1, 1]$ if f_0 syntomic

Remark: $\mathcal{W}_i$ lin. fund b/c

(a) $\mathcal{W}_i$ coh dim zero $\Rightarrow$ $\mathcal{W}_{i+1}$ coh dim zero (pf: long exact seq.)

(b) $\mathcal{W}_i$ coh dim zero + quot stack $\Rightarrow$ $\mathcal{W}_{i+1}$ quot stack

pf 1:

$$\begin{tikzcd} \mathcal{W}_i \arrow[r, hook, "\mathcal{J}"] \arrow[dr] & \mathcal{W}_{i+1} \arrow[d, dashed] \\ & BGL_n \end{tikzcd}$$

(Def problem 3)

$o \in \mathrm{Ext}^1(L_{BG}, \mathcal{J}) = 0$

$\parallel$

pf 2: $\mathcal{W}_i \xhookrightarrow{\mathcal{J}} \mathcal{W}_{i+1}$ $\quad o \in \mathrm{Ext}^2(\mathcal{E}_i, \mathcal{E}_i \otimes \mathcal{J})$

$\mathcal{E}_i$ v.b. $\exists \mathcal{E}_{i+1}$ v.b. $= 0$

3.4 Complete and henselian pairs

[AHR2, §3]

<u>Def</u>: $(\mathcal{X}, \mathcal{X}_0)$ pair of alg. stacks, $\mathcal{X}_0 \hookrightarrow \mathcal{X}$ closed immersion, is

- <u>local</u> if every closed non-empty subset of $|\mathcal{X}|$ intersects $|\mathcal{X}_0|$
- <u>henselian</u> if $\forall\, \mathcal{X}' \to \mathcal{X}$ finite:
$$\mathrm{Clopen}(\mathcal{X}') \to \mathrm{Clopen}(\mathcal{X}' \times_{\mathcal{X}} \mathcal{X}_0) \quad \text{bijective}$$
- <u>(coherently) complete</u> if $\mathcal{X}$ noetherian and
$$\mathrm{Coh}(\mathcal{X}) \to \varprojlim_{n \geq 0} \mathrm{Coh}(\mathcal{X}_n) \quad \text{equiv of categories}$$

where $\mathcal{X}_n$ n^{th} inf. nbhd of $\mathcal{X}_0$.

<u>Exc</u> 2.4 (easy) complete $\Rightarrow$ henselian $\Rightarrow$ local.

<u>Exc</u> 2.5 (not deep) $p: \mathcal{X}' \to \mathcal{X}$ proper. $(\mathcal{X}, \mathcal{X}_0)$ henselian $\Rightarrow$ $(\mathcal{X}', \mathcal{X}'_0)$ henselian

$\mathcal{X}'_0 := p^{-1}(\mathcal{X}_0)$

<u>Thm</u> (Grothendieck existence) "formal GAGA" —〃— $(\mathcal{X}, \mathcal{X}_0)$ affine complete $\Rightarrow$ $(\mathcal{X}', \mathcal{X}'_0)$ complete

<u>Thm</u> (easy) [AHR2, Thm 3.6] If
$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, hook, "\text{closed imm}"] \arrow[d, "\text{gms}"'] & \mathcal{X} \arrow[d, "\pi\ \text{gms}"] \\ X_0 \arrow[r, hook] & X \end{tikzcd}$$
$(X_0 = \pi(\mathcal{X}_0))$ then

$(\mathcal{X}, \mathcal{X}_0)$ henselian $\iff$ (X, X_0) henselian

<u>pf</u>: Since π univ closed w/ conn fibers:
$$\begin{tikzcd} \mathrm{Clopen}(X) \arrow[r, "\cong"] \arrow[d] & \mathrm{Clopen}(\mathcal{X}) \arrow[d] \\ \mathrm{Clopen}(X_0) \arrow[r, "\cong"] & \mathrm{Clopen}(\mathcal{X}_0) \end{tikzcd}$$

If $X' \to X$ finite, let $\mathcal{X}'_0 \hookrightarrow \mathcal{X}'$ be b.c. If $\mathcal{X}' \to \mathcal{X}$ finite, let $X' \to X$ gms $\Rightarrow$
$$\underbrace{\begin{tikzcd} \mathcal{X}'_0 \arrow[r, hook] \arrow[d, "\text{gms}"'] & \mathcal{X}' \arrow[d, "\text{gms}"] \\ X'_0 \arrow[r, hook] & X' \end{tikzcd}}_{\text{hens.}} \qquad \overbrace{\begin{tikzcd} \mathcal{X}'_0 \arrow[r, hook] \arrow[d, "\text{gms}"'] & \mathcal{X}' \arrow[d, "\text{gms}"] \\ X'_0 \arrow[r, hook] & X' \end{tikzcd}}^{\text{hens}} \quad \square$$

<u>Thm</u> (diff) [AHR2, 3.7, 5.1, 1.6] [AHR1, 1.3]
$$\begin{tikzcd} \mathcal{X}_0 \arrow[r, hook] \arrow[d, "\text{gms}"'] & \mathcal{X} \arrow[d, "\text{gms}"] \\ X_0 \arrow[r, hook] & X \end{tikzcd}$$
Assume X affine. $(\mathcal{X}, \mathcal{X}_0)$ complete $\overset{\text{easy}}{\Rightarrow}$ (X, X_0) complete

If $\mathcal{X}$ lin. fund e.g. $\mathcal{X} = [\operatorname{Spec} A / G]$ G lin red. (or more gen: $\Delta_{\mathcal{X}}$ aff $\mathcal{X}_0$ lin fund)

(X, X_0) complete $\overset{\text{"diff"}}{\Rightarrow}$ $(\mathcal{X}, \mathcal{X}_0)$ complete

3.5 Applications of henselian theorem HP

Thm LF (Luna's fundamental lemma)

$$\begin{tikzcd} x \in \mathcal{X} \arrow[r, "f"] \arrow[d, "\pi_{\mathcal{X}}"', "\text{gms}"] \arrow[dr, phantom, "\circ"] & \mathcal{Y} \arrow[d, "\text{gms}"] \\ X \arrow[r, "g"'] & Y \end{tikzcd}$$

s.th.

[AHR2, Thm 3.13]

(1) f étale and repr in nbhd of x

(2) x closed, $f(x)$ closed.

(3) $G_x \xrightarrow{\cong} G_{f(x)}$ iso

Then $\exists\, U \subset X$ open s.th.

$$\begin{tikzcd} x \in \pi_{\mathcal{X}}^{-1}(U) \subset \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d] \\ U \subset X \arrow[r] \arrow[r, bend right, "\text{étale}"'] & Y. \end{tikzcd}$$

<u>Idea of pf</u> (not diff.) WLOG $\mathcal{Y}$ henselian local strictly

$\Rightarrow (\mathcal{Y}, \mathcal{G}_y)$ henselian, $\mathcal{G}_x \cong \mathcal{G}_y$. Then use:

Prop: $(\mathcal{X}, \mathcal{X}_0)$ henselian $\Leftrightarrow$ $\forall\, \mathcal{X}' \to \mathcal{X}$ repr étale $\Gamma(\mathcal{X}'/\mathcal{X}) \xrightarrow{\cong} \Gamma(\mathcal{X}_0'/\mathcal{X}_0)$

to obtain a section s of f. If f sep, then $\operatorname{im}(s) \subset \mathcal{X}$ clopen and $U = \pi_{\mathcal{X}}(\operatorname{im}(s))$

In general, one extra step.

Thm (universality of good moduli spaces)

[AHR2, Thm 3.11]

$$\begin{tikzcd} \mathcal{X} \arrow[r] \arrow[d, "\pi"', "\text{gms}"] & Y \text{ alg space} \\ X \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

<u>pf</u>: WLOG X affine, Y qc. Pick $p: \operatorname{Spec} A = Y' \xrightarrow{\text{ét}} Y$.

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & Y' \arrow[d] \\ \mathcal{X} \arrow[r] & Y \end{tikzcd}$$

Pull-back to $\mathcal{X}' \xrightarrow{\text{ét}} \mathcal{X}$. Then étale-locally on X obtain

Since Y' affine $\Rightarrow$ $\mathcal{X} \to \operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \xrightarrow{\exists !} Y' \to Y$.

$$\operatorname{Spec} \Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = X$$

3.6 Examples of coh. complete stacks (assuming Thm CP)

Ex:
$$\begin{tikzcd} {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "gms"] \\ * \end{tikzcd} \Rightarrow [\mathbb{A}^1/\mathbb{G}_m] \text{ coh. complete along } 0$$

Ex:
$$\begin{tikzcd} \mathcal{X} \arrow[d, "gms"] \\ \mathrm{Spec}\, k \end{tikzcd} \Rightarrow \mathcal{X} \text{ coh complete along unique closed point.}$$

Ex:
$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"', "gms"] \\ X \end{tikzcd} \quad x \in |\mathcal{X}| \text{ closed point} \Rightarrow$$

coh complete along x

$$\begin{tikzcd} \mathcal{X} \times_X \hat{X}_{\pi(x)} \arrow[r] \arrow[d, "gms"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "gms"] \\ \hat{X}_{\pi(x)} \arrow[r] & X \end{tikzcd}$$

$\hat{X}_{\pi(x)} = \mathrm{Spec}\, \hat{A}$, $X = \mathrm{Spec}\, A$

$$\hat{\mathcal{X}}_x := \mathcal{X} \times_X \hat{X}_{\pi(x)}$$

"completion at x"

$$\mathrm{Coh}(\hat{\mathcal{X}}_x) = \varprojlim_{n \geq 0} \mathrm{Coh}(\mathcal{X}_x^{[n]})$$

3.7 Formal functions

Thm FF: $(\mathfrak{X}, \mathfrak{X}_0)$ noetherian (coh affine) w/ gms $(\operatorname{Spec} A, \operatorname{Spec} A/I)$ complete pair (i.e. A I-adic. complete $\Rightarrow A = \varprojlim A/I^n$). Then

[AHR1, Thm 1.3]
[AHR2, Cor 4.2]

$$\Gamma(\mathfrak{X}, \mathcal{F}) \xrightarrow{\cong} \varprojlim_n \Gamma(\mathfrak{X}, \mathcal{F}/\mathcal{I}^{n+1}\mathcal{F})$$

$\forall \mathcal{F} \in \mathrm{Coh}(\mathfrak{X})$

pf: Since $\pi : \mathfrak{X} \to \operatorname{Spec} A$ exact: RHS $= \varprojlim_n \Gamma(\mathfrak{X}, \mathcal{F}) / \Gamma(\mathfrak{X}, \mathcal{I}^n \mathcal{F})$

$F_n := \Gamma(\mathfrak{X}, \mathcal{I}^n\mathcal{F})$, $I_n := \Gamma(\mathfrak{X}, \mathcal{I}^n)$ are coherent. (GMS (g))

$\mathcal{I}^\bullet = \bigoplus_{n \geq 0} \mathcal{I}^n$ f.g. $\mathcal{O}_{\mathfrak{X}}$-alg, $\mathcal{I}^\bullet\mathcal{F} = \bigoplus_{n\geq 0} \mathcal{I}^n\mathcal{F}$ f.g. $\mathcal{I}^\bullet$-module

$$\begin{tikzcd} \operatorname{Spec}_{\mathfrak{X}} \mathcal{I}^\bullet \arrow[r, "\text{f.t.}"] \arrow[d, "\text{gms}"] & \mathfrak{X} \arrow[d, "\pi", "\text{f.t.}"'] \\ \operatorname{Spec} \bigoplus I_n \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

(GMS (f)) + (h)

$\Rightarrow$ $\bigoplus I_n$ finite type A-alg (GMS (i))
and $\bigoplus F_n$ f.g. $\bigoplus I_n$-mod (GMS (g))

$\bigoplus I_n$ not gen'd in deg 1, but $\exists N$: $I_{Nk} = (I_N)^k$ $\forall k$.
(suff div)
e.g. lcm of gen deg's

$\Rightarrow$ WLOG $I_n = I^n$.

$\bigoplus F_n$ not gen'd in deg 1, but $\exists N \gg 0$: $F_{n+1} = I F_n$ $\forall n \geq N$.
e.g. max of gen deg's

i.e. (F_n) I-stable filtration of F_0

Atiyah-Macdonald $\Longrightarrow$ $\varprojlim_{n\geq 0} F_0/F_n = \varprojlim_{n \geq 0} F_0/I^nF_0 \overset{\uparrow}{=} F_0$

b/c F_0 coherent A-module and A I-adic complete.

$\square$

3.8 Coherent completeness w/ resolution property

Thm CP [AHR1, 1.3] [AHR2, 5.1] $(\mathcal{X}, \mathcal{X}_0)$ lin fund w/ gms (X, X_0) complete, $X = \operatorname{Spec} A$.

Then $(\mathcal{X}, \mathcal{X}_0)$ complete, i.e.

$$\mathrm{Coh}(\mathcal{X}) \xrightarrow{\sim} \varprojlim_{n \geq 0} \mathrm{Coh}(\mathcal{X}^{[n]})$$

pf: Fully faithfulness: given $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(\mathcal{X})$, $Q = \mathcal{H}om(\mathcal{F}, \mathcal{G})$ amounts to

$$\Gamma(\mathcal{X}, Q) \xrightarrow{\cong} \varprojlim_n \Gamma(\mathcal{X}, Q/\mathcal{I}^n Q)$$

so follows from Thm FF.

Ess surj: Let $\{\mathcal{F}_n\} \in \varprojlim \mathrm{Coh}(\mathcal{X}^{[n]})$. Since $\mathcal{X}$ has res prop

$\exists \underset{\text{v.b.}}{\mathcal{E}} \overset{\phi_0}{\twoheadrightarrow} \mathcal{F}_0$. Can lift to $\mathcal{E} \xrightarrow{\phi_n} \mathcal{F}_n$ $\forall n$ b/c

$$\twoheadrightarrow \mathcal{E}^\vee \otimes \mathcal{F}_n \twoheadrightarrow \mathcal{E}^\vee \otimes \mathcal{F}_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow \mathcal{E}^\vee \otimes \mathcal{F}_0$$

Γ exact $\Rightarrow$

$$\begin{array}{ccccc} \Gamma(\mathcal{X}, \mathcal{E}^\vee \otimes \mathcal{F}_n) & \twoheadrightarrow & \Gamma(\mathcal{X}, \mathcal{E}^\vee \otimes \mathcal{F}_{n-1}) \twoheadrightarrow & \cdots \twoheadrightarrow & \Gamma(\mathcal{X}, \mathcal{E}^\vee \otimes \mathcal{F}_0) \\ \| & & \| & & \| \\ \mathrm{Hom}(\mathcal{E}, \mathcal{F}_n) & \twoheadrightarrow & \mathrm{Hom}(\mathcal{E}, \mathcal{F}_{n-1}) \twoheadrightarrow & \cdots \twoheadrightarrow & \mathrm{Hom}(\mathcal{E}, \mathcal{F}_0) \end{array}$$

$(\mathcal{X}, \mathcal{X}_0)$ local $\overset{\text{Nakayama}}{\Longrightarrow}$ $\mathcal{E} \xrightarrow{\phi_n} \mathcal{F}_n$ surj.

Repeat arg on $\ker\{\phi_n\}$ gives $\mathcal{E}'$ and $\mathcal{E}'_n \xrightarrow{\psi_n} \mathcal{E}_n$ s.th. $\mathcal{F}_n = \operatorname{coker}(\psi_n)$

~~Thm FF~~ Fully faithfulness gives $\mathcal{E}' \xrightarrow{\psi} \mathcal{E}$ and $\mathcal{F} := \operatorname{coker}(\psi)$

maps to $\{\mathcal{F}_n\}$. $\square$