LECTURE 2: Local structure of algebraic stacks

2.1 When is a stack a quotient stack?

$$\mathcal{X} = [U/G] \quad \Rightarrow \quad \begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d] \\ \mathcal{X} \arrow[r] & BG \end{tikzcd}$$

U alg. space $\Leftrightarrow$ $\mathcal{X} \to BG$ representable (by alg. spaces)

$\Leftrightarrow \forall x \in |\mathcal{X}| \quad \underset{\substack{\uparrow \\ \text{(welldefined up to conj.)}}}{G_x \longrightarrow G}$ injective

G affine$/k$ $\Rightarrow \exists\, G \hookrightarrow GL_n$

$$\begin{tikzcd} U \overset{G}{\times} GL_n \arrow[r] \arrow[d] & GL_n/G \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\ \mathcal{X} \arrow[r] & BG \arrow[r] & BGL_n \end{tikzcd}$$

Def: $\mathcal{X}$ quotient stack $\overset{\text{def}}{\Leftrightarrow} \underset{U \text{ alg space}}{\mathcal{X} = [U/GL_n]} \Leftrightarrow \underset{V \text{ alg space}}{\mathcal{X} = [V/G]}$ G affine

$\Leftrightarrow \exists\, \mathcal{E}$ v.b. $\forall x \in |\mathcal{X}|$ G_x acts faithfully on $\mathcal{E}_x$.

Def: $\mathcal{X}$ has resolution property if $\forall\, \mathcal{F} \in QCoh_{f.t.}(\mathcal{X})$ $\exists$ v.b. $\mathcal{E}$ and $\mathcal{E} \twoheadrightarrow \mathcal{F}$.

Thm (Totaro '04, Gross '17) $\mathcal{X}$ res prop + aff. stab + qcqs

$\Leftrightarrow \mathcal{X} = [\text{quasi-affine}/GL_n]$

2.2 Linearly fundamental stacks

$\mathcal{X}$ qcqs, cohomologically affine (i.e. Γ exact)

$\Leftrightarrow$ $\mathcal{X} \longrightarrow \mathrm{Spec}(\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}))$ good moduli space

Lemma: Let $\mathcal{X}$ as above TFAE

$\mathcal{X}$ quot stack ~~(+ aff. stab)~~ (part of def) $\Leftrightarrow$ $\mathcal{X}$ res prop + aff stab $\Leftrightarrow$ $\mathcal{X} = [\text{affine}/GL_n]$

pf: $\Leftarrow$ obvious. If $\mathcal{X}$ quot stack, then $\mathcal{X} = [U/GL_n]$ with U alg space ~~for U affine~~

$U \longrightarrow \mathcal{X}$ affine $\Rightarrow$ U coh affine $\underset{\text{Serre's theorem}}{\Longrightarrow}$ U affine

Def: $\mathcal{X}$ fundamental if $\mathcal{X} = [\text{affine}/GL_n]$

$\mathcal{X}$ lin fundamental if —— " —— and $\mathcal{X}$ cohomologically aff

Rmk: fund $\Leftrightarrow$ lin fund in char 0

In general, fund $\Rightarrow$ adequately affine.

2.3 Local structure for schemes and algebraic spaces

X scheme

$x \in X$

$$\begin{tikzcd}
 & \mathrm{Spec}\,A = U \ni u \arrow[dl, dashed, "\text{étale}", "g"'] \arrow[dr, "\text{open imm}", "f"'] & \\
\mathbb{A}^n & & X \\
 & u \arrow[r, mapsto] & x
\end{tikzcd}$$

$\underbrace{\qquad\qquad}_{\text{if } X \text{ smooth at } x}$

(can take $g(u) = 0$ if $k(x)$ sep. over k)

X alg space

$x \in X$

$$\begin{tikzcd}
 & \mathrm{Spec}\,A = U \ni u \arrow[dl, dashed, "\text{étale}", "g"'] \arrow[dr, "\text{étale}", "f"'] & \\
\mathbb{A}^n & & x \in X \\
 & u \arrow[r, mapsto] & x
\end{tikzcd}$$

$\underbrace{\qquad\qquad}_{\text{if } X \text{ smooth at } x}$

— " —

$k(u) = k(x)$

Thm (Artin '69) X, Y finite type over excellent scheme (e.g. field)

$x \in X$, $y \in Y$ $\exists\, \hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{Y,y} \iff \exists$

$$\begin{tikzcd}
 & (U,u) \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\
(X,x) & & (Y,y)
\end{tikzcd}$$

w/ $k(x) = k(u) = k(y)$

pf Artin approximation on X:

$$\begin{tikzcd}
 & \mathrm{Spec}\,\hat{\mathcal{O}}_{X,x} \arrow[dl] \arrow[dr] & \\
X & & Y
\end{tikzcd}
\rightsquigarrow
\begin{tikzcd}
 & \mathrm{Spec}\,\hat{\mathcal{O}}_{X,x} \arrow[d] \arrow[ddr] & \\
 & X' \arrow[dl, "\text{étale}"'] \arrow[dr, "\text{étale}"] & \\
X & & Y
\end{tikzcd}$$

not commutative but up to m_x^2

$\Rightarrow \hat{\mathcal{O}}_{X',x'} \cong \hat{\mathcal{O}}_{Y,y}$

2.4 Local structure for DM-stacks

Thm ("Keel-Mori") ($k = \bar{k}$ for simplicity) $\mathcal{X}$ DM-stack$/k$, $x \in \mathcal{X}(k)$

Then $\exists\, U$ affine

$$\begin{tikzcd} & {[U/G_x]} \arrow[dl, dashed, "g"', "\text{ét}"] \arrow[dr, "f", "\text{étale}"'] & \\ {[\mathbb{A}^n/G_x]} & u \arrow[r, mapsto] & x \in \mathcal{X} \end{tikzcd}$$

$u \in [U/G_x]$ $\qquad$ $G_u = G_x$

$\underbrace{[\mathbb{A}^n/G_x]}$ if $\mathcal{X}$ smooth at x and G_x lin red.

G_x acts lin. on $\mathbb{A}^n$ = tangent space at x.

If $\Delta_{\mathcal{X}}$ sep, then f repr + sep.

If $I_{\mathcal{X}}$ finite, then f stabilizer-preserving. (this is used to construct $\mathcal{X}_{cms}$)

Cor: If $I_{\mathcal{X}}$ finite ($\Rightarrow \exists\, \mathcal{X}_{cms}$) then:

$$\begin{tikzcd} {[\mathbb{A}^n/G_x]} \arrow[d] \arrow[dr, phantom, "\square"] & {[U/G_x]} \arrow[l, dashed, "g"'] \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ \mathbb{A}^n/G_x & U/G_x \arrow[l, dashed] \arrow[r] & \mathcal{X}_{cms} \end{tikzcd}$$

$\underbrace{\mathbb{A}^n/G_x}$ if $\mathcal{X}$ smooth at x

<u>pf of Thm</u>: Pick an étale pres $V \xrightarrow{p} \mathcal{X}$ and use ~~(variant of~~) $\mathrm{Hilb}(V/\mathcal{X})$. to reduce to p finite étale. Then to Σ_d and then G_x (see Exercise 2.1)

<u>pf of Cor</u>: Luna's fund. lemma (later in general)

2.5 Local structure of Artin stacks

[AHR1, Thms 1.1, 1.2] + [AHR2, Thm 1.1]

Thm LS ($k = \bar{k}$ for simplicity) $\mathcal{X}$ alg stack f.p./k, $x \in \mathcal{X}(k)$. Assume

(1) $\mathcal{X}$ has affine stabilizers.

(2) G_x ~~has~~ lin. reductive.

Then $\exists$

$$\begin{tikzcd} & \mathcal{W} \arrow[dl, dashed, "\text{ét}"', "g"] \arrow[dr, "\text{étale}", "f"'] & \\ \underbrace{[\mathbb{A}^n/G_x]}_{\text{if } \mathcal{X} \text{ smooth at } x} & & \mathcal{X} \end{tikzcd}$$

$w \in \mathcal{W}$, $w \mapsto x \in \mathcal{X}$

$\mathcal{W} = [\operatorname{Spec} A/G_x]$

$\Downarrow$

w closed pt.

f stab. pres. at w.

(a) If $\Delta_{\mathcal{X}}$ affine, then f affine.

(b) If $\Delta_{\mathcal{X}}$ sep, then f repr.

(c) If $\mathcal{X}$ has gms, then
$$\begin{tikzcd} \mathcal{W} \arrow[r, "\text{ét}"] \arrow[d, "\text{gms}"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\text{gms}"] \\ W \arrow[r, "\text{ét}"'] & X \end{tikzcd}$$
. (and always sim. for g)

Generalizations

(i) Given $H \subset G_x$ lin. red. $\Rightarrow$
$$\begin{tikzcd} \mathcal{W} = [\operatorname{Spec} A/H] \arrow[r, "f"] \arrow[dr, phantom, "\square"] & \mathcal{X} \\ BH \arrow[u, hook] \arrow[r] & BG_x \arrow[u, hook] \end{tikzcd}$$

G_x/H étale smooth syntomic $\Rightarrow$ f étale smooth syntomic [AHHR3]

(ii) over arb base [AHR2]

given $\mathcal{W}_0 \xrightarrow{f_0} \mathcal{G}_x$ lin fund gerbe $\Rightarrow$
$$\begin{tikzcd} \mathcal{W} = [\operatorname{Spec} A/GL_n] \arrow[r, "f"] \arrow[dr, phantom, "\square"] & \mathcal{X} \\ \mathcal{W}_0 \arrow[u, hook] \arrow[r, "f_0"] & \mathcal{G}_x \arrow[u, hook] \end{tikzcd}$$

f_0 ét/sm/synt $\Rightarrow$ f ét/sm/synt

(iii) [AHHR3]

given $\mathcal{W}_0 \xrightarrow{f_0} \mathcal{X}_0$ ét/sm/synt $\Rightarrow$ as above
lin fund

(iv) $\mathcal{X}$ not nec f.p./base.

2.6 Examples of local structure theorem

Known case:

① $\mathcal{X} = [U/G]$ U affine, G lin red $\overset{\text{Luna's étale slice}}{\Longrightarrow}$ $\overset{[Z/G_x]}{\overset{\parallel}{\mathcal{W}}} \xrightarrow{\text{ét}} \mathcal{X}$

where $Z \hookrightarrow U$ loc closed

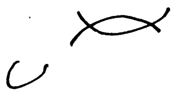
also works when U normal scheme, G ~~arbitrary~~ aff smooth, G_x lin red using Sumihiro [AK16, Rmk 3.7]

② Another Sumihiro result: $\mathcal{X} = [U/T]$ U normal scheme, T torus, $x \in U$.

$\Rightarrow \exists\ \overset{}{\underset{[V/T]}{\underset{\parallel}{\mathcal{W}}}} \xrightarrow{\text{open}} \mathcal{X}$ $\quad V \subset U$ aff. open nbhd of x

③ $\mathcal{X} = [\underset{\text{proj cusp.}}{\propto} / \mathbb{G}_m]$ node has no open affine $\mathbb{G}_m$-equi nbhds.

[AHR1, Ex 1.6]



$\xrightarrow[2:1]{\text{fét}} \propto$ $\quad \mathbb{G}_m$-equi

$= + = \operatorname{Spec} k[x,y]/(xy)$

$\mathcal{W} = [+/\mathbb{G}_m] \xrightarrow{\text{ét}} [\propto/\mathbb{G}_m]$

④ $\mathcal{L}og$ parametrizes log structures. A k-point is a f.g. integral monoid M.

$\Rightarrow$ log structure $M \oplus k^\times \to k$, $\quad m, \lambda \mapsto \begin{cases} \lambda & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}$

[AHR1, Ex 1.8] Ex: $\operatorname{Aut}(\mathbb{N}^r \oplus k^\times) = \mathbb{G}_m^r \rtimes \Sigma_r$

Local structure $\begin{array}{ccc} [\operatorname{Spec} k[M]/M^{gp}] & \longrightarrow & \mathcal{L}og \\ \Psi & & \Psi \\ 0 & \longmapsto & [M] \end{array}$ repr. ét not stab. pres

Ex: $[\mathbb{A}^r/\mathbb{G}_m^r] \longrightarrow \mathcal{L}og$ repr. + ét.

$B\mathbb{G}_m^r \longrightarrow B\mathbb{G}_m^r \rtimes S_r$

$[\mathbb{A}^r/\mathbb{G}_m^r \rtimes \Sigma_r] \longrightarrow \mathcal{L}og$ ét, stab pres at 0 not repr. (gen: $B\Sigma_r \to *$)

Reason: $\Delta_{\mathcal{L}og}$ not separated!

Counter-examples

⑤ $\exists\, G \longrightarrow \mathbb{A}^1$ gp scheme w/ generic fiber E elliptic curve
special fiber $\mathbb{G}_m$.

[AHR1, Ex 1.5]

$\mathcal{X} = BG$. $\nexists\, [\operatorname{Spec} A/\mathbb{G}_m] \xrightarrow{\text{étale}} BG$

NB! BG does not have aff stab.

⑥ $\exists\, \underset{\operatorname{Spec} k[x,y]_{xy+1}}{\underset{\shortparallel}{G}} \longrightarrow \underset{\operatorname{Spec} k[x]}{\underset{\shortparallel}{\mathbb{A}^1}}$ gp scheme w/ generic fiber $\mathbb{G}_m$
special fiber $\mathbb{G}_a$

[AHR1, Ex 1.4]

$\mathcal{X} = BG$ $\nexists\, [\operatorname{Spec} A/\mathbb{G}_a] \xrightarrow{\text{étale}} BG$

NB! $\mathbb{G}_a$ not (lin) red.

2.7 Outline of proof

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, "f_0"] \arrow[d, phantom, "\shortparallel"] & \mathcal{X}_0 \arrow[d, phantom, "\shortparallel"] \\ (BH) & (BG_x) \end{tikzcd}$$

ét/smooth (or synt.)

or lin fund

Def: $\mathcal{X}^{[n]}$ = nth inf nbhd of $\mathcal{X}_0 = V(m_x^{n+1}) \hookrightarrow \mathcal{X}$

Step 1: $\exists$

$$\begin{tikzcd} \mathcal{W}_0 \arrow[r, hook] \arrow[d, "f_0"'] \arrow[dr, phantom, "\square"] & \mathcal{W}_1 \arrow[r, hook] \arrow[d, "f_1"] \arrow[dr, phantom, "\square"] & \mathcal{W}_2 \arrow[r, hook] \arrow[d, "f_2"] \arrow[dr, phantom, "\square"] & \cdots \\ \mathcal{X}^{[0]} \arrow[r, hook] & \mathcal{X}^{[1]} \arrow[r, hook] & \mathcal{X}^{[2]} \arrow[r, hook] & \cdots \end{tikzcd}$$

f_i ét/smooth

(infinitesimal) deformation theory (trivial step if $\mathcal{W}_0 = \mathcal{X}_0$)

Step 2 $\exists\ \widehat{\mathcal{W}} = [\operatorname{Spec} B/H]$ (or lin fund) s.th. $\mathcal{W}_0 \overset{\text{closed}}{\hookrightarrow} \widehat{\mathcal{W}}$, $\mathcal{W}_n = \mathcal{W}^{[n]}$

$\operatorname{Coh}(\widehat{\mathcal{W}}) \xrightarrow{\sim} \varprojlim \operatorname{Coh}(\mathcal{W}_n)$

Coherent completeness (Thm CP) and effectivity + Thm E

Step 3: Tannaka duality (Thm TD) $\Rightarrow \widehat{\mathcal{W}} = \varinjlim \mathcal{W}_n \Rightarrow \exists\ \widehat{\mathcal{W}} \xrightarrow{\hat{f}} \mathcal{X}$ extending $\{f_n\}$

Step 4: Artin approximation (Thm AA) (if $\mathcal{X}$ smooth)

or equiv Artin algebraization (general) (Thm EAA)

$$\Rightarrow \exists\ \begin{tikzcd} \widehat{\mathcal{W}} \arrow[r] \arrow[rr, bend left, "\hat{f}"] & \mathcal{W} \arrow[r, "f"] \arrow[d, phantom, "\shortparallel"] & \mathcal{X} \\ & {[\operatorname{Spec} A/H]} & \end{tikzcd}$$

f.t/k

f ét/smooth

Refinements

(a) fairly easy using coh. aff.

(b) using a result of [AHH19]

(c) follows from Luna's fundamental lemma. (Thm LF)