LECTURE 1: Quotient stacks and good moduli spaces

1.1 Groupoids

1.2 Algebraic stacks and presentations

1.3 Quotient stacks

1.4 Topological space and stab. groups and inertia stack

1.5 More quotient stacks (explicit examples)

Dictionary

1.6 Aside: examples from moduli
- $\mathcal{M}_g, \ldots$ $\mathrm{Hom}(\mathfrak{X}, \mathcal{Y})$
- $\mathrm{Coh}_{X/S}, \ldots$ $\mathrm{Hom}(\mathcal{F}, \mathcal{G})$
- $\mathcal{L}og$

1.7 Coarse moduli spaces

1.8 Good moduli spaces: def + props

1.9 Examples of good moduli spaces

1.10 GIT and good moduli spaces

1.11 Example GIT (4 pts on $\mathbb{P}^1$)

§1.1 Groupoids

<u>Def</u>: A groupoid $\mathcal{X}$ is a $(1,0)$-category: $\left(\text{1-category where all 1-morphisms (=arrows) are invertible}\right)$

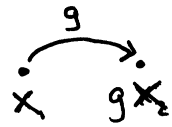
$$\underset{\circlearrowleft\, i}{Ar(\mathcal{X})} \;\underset{t}{\overset{s}{\rightrightarrows}}\; ob(\mathcal{X}) \quad (e : ob(\mathcal{X}) \to Ar(\mathcal{X})) \qquad = \Big[\underset{\text{set}}{Ar(\mathcal{X})} \rightrightarrows \underset{\text{set}}{ob(\mathcal{X})}\Big]$$

<u>Def</u>: A 1-morphism of grpds $=$ functor $\mathcal{X} \xrightarrow{f} \mathcal{Y}$

2-morphism $=$ nat. tfm. $f_1 \Rightarrow f_2$ <u>Exc</u>: always nat. <u>iso</u>

(Grpd) is a $(2,1)$-category $\left(\text{2-category where all 2-morphisms inv.}\right)$

<u>Ex</u>: G discrete group. $BG = [\,G \rightrightarrows *\,] =$ 

<u>Ex</u>: $G \circlearrowright X$ set. $[X/G] = \Big[\,G \times X \underset{\pi_2}{\overset{\sigma}{\rightrightarrows}} X\,\Big] =$ 

<u>Exercises</u>:

<u>Exc 1.1</u>: (a) $\mathrm{Hom}_{\mathrm{Grpd}}(BG, BH) = \big[\,\mathrm{Hom}_{\mathrm{Grp}}(G,H)\,/\,H\,\big]$

H acts via conjugation.

(b) 2-fiber product in (Grpd)

$$\begin{tikzcd} \mathcal{X} \times_{\mathcal{Z}} \mathcal{Y} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{Y} \arrow[d, "g"] \\ \mathcal{X} \arrow[r, "f"'] & \mathcal{Z} \end{tikzcd}$$

$\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ groupoid with

<u>objs</u>: (x, γ, y) $x \in ob\,\mathcal{X}$, $y \in ob\,\mathcal{Y}$, $\gamma : f(x) \to g(y)$

<u>mor</u>:

(c) Exact seq of groups $1 \to H \to G \to G/H \to 1$

$\rightsquigarrow$

~~2G fiber~~
2-cartesian diagram

$$\begin{tikzcd}
H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] & \\
G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G/H \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\
* \arrow[r] & BH \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & BG \arrow[d] \\
& * \arrow[r] & B(G/H)
\end{tikzcd}$$

(b')
$$\begin{tikzcd}
G \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d] \\
* \arrow[r] & BG
\end{tikzcd}$$

(b'') $Ob(\mathfrak{X}) \times_{\mathfrak{X}} Ob(\mathfrak{X}) = Ar(\mathfrak{X})$ (set)

Exc 1.2 Describe $BG = [BH/(G/H)]$!

Exc 1.3 Kummer seq. $1 \to \mu_n \to \mathbb{G}_m \to \mathbb{G}_m \to 1$

Describe action of $\mathbb{G}_m$ on $B\mu_n$.

§1.2 Algebraic stacks and presentations

Schemes

either ① $(X, \mathcal{O}_X)$ loc. ringed space

or ② $X : (\mathrm{AffSch})^{op} \to (\mathrm{Set})$ Zariski-sheaf

s.th. $\exists$ Zariski atlas = presentation $\quad U := \coprod \operatorname{Spec} A_i \xrightarrow{\coprod p_i} X$ surjective

p_i represented by open immersions

$$\left(\begin{tikzcd} \operatorname{Spec} A_i \arrow[r, "p_i"] & X \\ V \arrow[u] \arrow[r, "\text{open imm}"'] \arrow[ur, phantom, "\square"] & \operatorname{Spec} B \arrow[u, "\forall"'] \end{tikzcd}\right)$$

repr by sch.

Alg spaces

① $(X_{\text{ét}}, \mathcal{O}_{X_{\text{ét}}})$ loc. ringed topos

② $X : (\mathrm{AffSch})^{op} \to (\mathrm{Set})$ étale sheaf

s.th. $\exists$ étale atlas = pres $\quad \coprod p_i$, p_i repr by étale $\qquad \underset{\text{scheme}}{U} \xrightarrow{p} X$ étale surj

Alg stacks

① only for DM-stacks

② $\mathcal{X} : (\mathrm{AffSch})^{op} \to (\mathrm{Grpd})$ étale stack

s.th. $\exists$ smooth atlas $p : \underset{\text{scheme}}{U} \to \mathcal{X}$ (repr. by smooth maps of alg spaces)

Rmk: Replacing étale/smooth w/ fppf/fppf gives equiv notion

Def: $\mathcal{X}$ DM if $\exists$ étale atlas. (Prop: $\mathcal{X}$ DM $\Leftrightarrow$ $\Delta_{\mathcal{X}}$ unramified

$\Leftrightarrow$ stab. gps finite étale)

Rmk: $\left(\begin{array}{c} \text{Alg. stacks} \\ \text{w/ given pres} \\ U \to \mathcal{X} \end{array}\right) \longleftrightarrow \left(\begin{array}{c} \text{Smooth groupoids} \\ \text{in algebraic spaces} \\ (R \rightrightarrows U) \end{array}\right)$

$$U \to \mathcal{X} \longmapsto R = U \times_{\mathcal{X}} U \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} U$$

$$U \to [U/R] \longleftarrow\!\shortmid (R \rightrightarrows U)$$

§1.3 Examples of alg. stacks: quotient stacks

Ex 1: G group scheme (over some base S) $\overset{\text{flat + f.p.}}{}$ $\circlearrowright X$ scheme/alg sp.

$\rightsquigarrow$ action groupoid $G \times X \underset{\pi_2}{\overset{\sigma}{\rightrightarrows}} X$ and stack quotient $[X/G] = [G \times X \rightrightarrows X]$ (in fppf top. if G not smooth)

Features:

(1) $X \xrightarrow{p} [X/G]$ G-torsor

$$\begin{tikzcd} G \times X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ X \arrow[r, "p"'] & {[X/G]} \end{tikzcd}$$

In part, p flat + surj.

(2) X smooth $\Rightarrow$ $[X/G]$ smooth.
red ⋮ red ⋮

(3) Functor of points: a T-point of $[X/G]$ is:

$$\begin{tikzcd} E \arrow[r, "G\text{-equi}"] \arrow[d, "G\text{-torsor}"'] & X \\ T \end{tikzcd}$$

$$\begin{tikzcd} E \arrow[r, "G\text{-equi}"] \arrow[d, "G\text{-torsor}"'] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ T \arrow[r] & {[X/G]} \end{tikzcd}$$

Ex: $\mu_2 \circlearrowright \mathbb{A}^2$ w/ weights $(1,1)$. $\mathbb{A}^2 \xrightarrow[\text{étale}]{2:1} \overset{\text{smooth}}{[\mathbb{A}^2/\mu_2]} \to \mathbb{A}^2/\mu_2 = k[u,v,w]/(uv-w^2)$ singular

Ex 2: G as above. $BG = [S/G]$, $BG(T) = \left\{ \begin{array}{c} E \\ \downarrow \\ T \end{array} G\text{-torsor} \right\}$

To be precise: $BG(T)$ groupoid w/ objects: $\begin{array}{c} E \\ \downarrow \\ T \end{array}$ G-torsor

morphisms: $E_1 \xrightarrow[G\text{-eq.}]{\cong} E_2$ over T, commuting ($\circ$)

$$\begin{tikzcd} E \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ T \arrow[r, "{[E]}"'] & BG \end{tikzcd}$$

Rmk:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & S \arrow[d, "\text{univ } G\text{-torsor}"] \\ {[X/G]} \arrow[r] & BG \end{tikzcd}$$

§1.4 "Underlying" topological space, stabilizer groups and inertia

$\mathcal{X}$ alg. stack.

<u>Def</u>: $|\mathcal{X}| = \{ \operatorname{Spec} k \xrightarrow{x} \mathcal{X} \} / \sim$.
for some field k

Here $x_1 \sim x_2$ if $\exists$ $\operatorname{Spec} K$

$$\begin{tikzcd} & \operatorname{Spec} k_1 \arrow[dr] & \\ \operatorname{Spec} K \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circ"] & & \mathcal{X} \\ & \operatorname{Spec} k_2 \arrow[ur] & \end{tikzcd}$$

<u>Rmk</u>: Functorial in $\mathcal{X}$. Equip w/ finest topology s.t. $|U| \to |\mathcal{X}|$ continuous for a presentation $U \xrightarrow{p} \mathcal{X}$.

<u>Stabilizer group</u>: $\operatorname{Spec} k \xrightarrow{x} \mathcal{X}$ gives $G_x \to \operatorname{Spec} k$ group scheme

and $BG_x \to \mathcal{X}$. If $\mathcal{X}/\bar{k}$ and $\operatorname{Spec} \bar{k} \to \mathcal{X}$, then $BG_x \xhookrightarrow{\text{mono}} \mathcal{X}$.

$$\begin{tikzcd} BG_x \arrow[r] & \mathcal{X} \\ \operatorname{Spec} k \arrow[u] \arrow[ur, "x"'] \arrow[r, phantom, "\circ"] & \end{tikzcd}$$

In general: $\operatorname{Spec} k \to \mathcal{G}_x \xhookrightarrow{\text{mono}} \mathcal{X}$ $\qquad$ $\mathcal{G}_x$ <u>residual gerbe</u>.
($\mathcal{X}$ qsep) $\qquad \mathcal{G}_x \downarrow \text{gerbe}$ $\operatorname{Spec} k(x)$

So $|\mathcal{X}| = \{ \mathcal{G}_x \hookrightarrow \mathcal{X} \}$ (no equiv)

<u>Inertia stack</u>: $T \xrightarrow{x} \mathcal{X}$ gives group $\operatorname{Aut}_{\mathcal{X}(T)}(x) = \left\{ \begin{tikzcd} T \arrow[r, bend left, "x"] \arrow[r, bend right, "x"'] \arrow[r, phantom, "\Downarrow\gamma"] & \mathcal{X} \end{tikzcd} \right\}$

gives (<u>non-flat</u>) group $\underset{\text{alg space}}{\text{scheme}}$ $\operatorname{Aut}(x) \to T$.

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \\ \operatorname{Aut}(x) \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & T \arrow[u, "{(x,x)}"'] \end{tikzcd}$$

Universal Aut:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \\ I_{\mathcal{X}} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathcal{X} \arrow[u, "\Delta"'] \end{tikzcd}$$

Inertia stack $I_{\mathcal{X}} \to \mathcal{X}$ relative group alg. space.

§1.5 More examples of quotient stacks and dictionary

$|[X/G]| = |X|/|G|$ orbit space

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[rr, bend right, "\bar{x}"'] & X \arrow[r] & {[X/G]} \end{tikzcd}$$

$G_{\bar{x}} = G_x$ stabilizer

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^1$, $\lambda . t = \lambda t$ $\quad [\mathbb{A}^1/\mathbb{G}_m] = $ ($B\mathbb{G}_m$ $\quad *$)

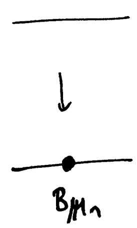
closed pt $\quad$ open pt

$\lambda . t = \lambda^n t$ $\quad [\mathbb{A}^1/\mathbb{G}_m] = $ ($B\mathbb{G}_m$ $\quad B\mu_n$)

closed $\quad$ open

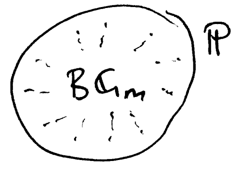
Ex: $\mu_n \curvearrowright \mathbb{A}^1$ $\quad \lambda . t = \lambda t$

$$\begin{tikzcd} \mathbb{A}^1 \arrow[d] \\ {[\mathbb{A}^1/\mu_n]} \end{tikzcd}$$



DM-stack (if n invertible)

Ex: $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\quad \lambda . (x,y) = (\lambda x, \lambda y)$ $\quad [\mathbb{A}^2/\mathbb{G}_m] = $



unique closed pt $B\mathbb{G}_m$ (= image of 0)

Cartesian diagrams:

$$\begin{tikzcd} G \times X \arrow[r, "{(\sigma, \pi_2)}"] \arrow[d] \arrow[dr, phantom, "\square"] & X \times X \arrow[d, "p \times p"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

$$\begin{tikzcd} G \times X \arrow[r, "\pi_2"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ X \arrow[r, "p"] & \mathcal{X} \end{tikzcd}$$

" $Ob(\mathcal{X}) = X$
$Ar(\mathcal{X}) = G \times X$ "

$\mathcal{X} = [G \times X \rightrightarrows X]$

$$\begin{tikzcd} \mathrm{Stab}(X) \arrow[r] \arrow[d, hook] \arrow[dr, phantom, "\square"] & X \arrow[d, hook, "\Delta"] \\ G \times X \arrow[r] & X \times X \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Stab}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

Dictionary

Equivariant geometry

$G \circlearrowright X$

G-orbit $Gx = p^{-1}(y)$

stabilizer G_x

$p^{-1}(\mathcal{Z}) = Z \hookrightarrow X$ G-inv. (closed / open) subsch

$p^*\mathcal{G} = \mathcal{F}$ qcoh $\mathcal{O}_X$-mod w/ G-action

$H^i_G(X, \mathcal{F})$

$\Gamma(X, \mathcal{F})^G$

$$\begin{tikzcd} G \times X \arrow[r] & X \times X \\ \operatorname{Stab} X \arrow[u, hook] \arrow[r] \arrow[ur, phantom, "\square"] & X \arrow[u, hook] \end{tikzcd}$$

Stacks

$\mathcal{X} = [X/G]$

point $y \in |\mathcal{X}|$

stab. G_y

$\mathcal{Z} \hookrightarrow \mathcal{X}$ (closed / open) substack

$\mathcal{G}$ qcoh $\mathcal{O}_{\mathcal{X}}$-mod

$H^i(\mathcal{X}, \mathcal{G})$

$\Gamma(\mathcal{X}, \mathcal{G})$

$$\begin{tikzcd} \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \\ I_{\mathcal{X}} \arrow[u] \arrow[r] \arrow[ur, phantom, "\square"] & \mathcal{X} \arrow[u] \end{tikzcd}$$

$\mathcal{X} = [X/G]$

$I_{\mathcal{X}} = [\operatorname{Stab} X / G]$ (conj. action)

Ex: $\mathcal{X} = BG$, $I_{\mathcal{X}} = [G/G]$ (conj action)

§1.6 Examples from moduli

(a) $\mathcal{M}_g : (\mathcal{S}ch)^{op} \to (\mathrm{Grpd})$

$$T \longmapsto \left\{ \begin{matrix} C \\ \downarrow \\ T \end{matrix} \ \begin{matrix} \text{proj, conn.} \\ \text{flat family of} \\ \text{smooth curves} \\ \text{of genus } g \end{matrix} \right\} \qquad \underline{\text{mor:}} \quad \begin{tikzcd} C_1 \arrow[rr, "\sim"] \arrow[dr] & \circ & C_2 \arrow[dl] \\ & T & \end{tikzcd}$$

<u>Fact</u>: $\mathcal{M}_g$ smooth irr alg. stack.

$\mathrm{Spec}\, k \xrightarrow{x} \mathcal{M}_g \quad \leftrightarrow \quad C/k$ curve, $\mathrm{Aut}(x) = \mathrm{Aut}(C)$.

<u>Ex</u>: $\mathcal{M}_0 = B PGL_2$ $\qquad \dim \mathcal{M}_g = 3g-3$.

$\mathcal{M}_g$ DM-stack for $g \geq 2$.

<u>Variants</u>
$\overline{\mathcal{M}}_{g,n}$, $\overline{\mathcal{M}}_{g,n}(X)$
$\mathfrak{M}$
polarized varieties
surfaces of gen type
K3-surfaces

(b) $\begin{tikzcd} \mathcal{X} \arrow[dr, "\text{flat proper}"'] & & \mathcal{Y} \arrow[dl] \\ & S & \end{tikzcd}$ $\quad \mathrm{Hom}(\mathcal{X}, \mathcal{Y})$

(c) $X \xrightarrow{f} S$ (flat) proper, fin. pres. $\qquad$ (and $\mathrm{Hilb}_{X/S}$, $\mathrm{Quot}_{\mathcal{F}/X/S}$)

$$\mathrm{Coh}_{X/S} : (\mathcal{S}ch/S)^{op} \to (\mathrm{Grpd})$$
$$(T \to S) \longmapsto \{ \mathcal{F} \in \mathrm{Coh}(X \times_S T) \text{ f.p., flat}/T \}$$

If f <u>flat</u>:
$VB_{n, X/S} \subset \mathrm{Coh}_{X/S}$ substack of $\mathcal{F}$ loc free of rank n. (vector bundles)

$\mathcal{P}ic_{X/S} = VB_{1, X/S}$ ——— " ——— 1 (line bundles)

Can be <u>rigidified</u> to alg. space if f coh. flat in dim zero.

(d) $\mathrm{Hom}(\mathcal{F}, \mathcal{G}) \supset \mathrm{Isom}(\mathcal{F}, \mathcal{G})$
(diagonal of $\mathrm{Coh}_{X/S}$)

(e) $\mathcal{L}og$, $\mathcal{L}og^{al}$ moduli of (aligned) log structures

§1.7 ~~Goarse~~ Coarse moduli spaces

<u>Def</u>: A <u>coarse moduli space</u> is a map $\underset{\text{alg stack}}{\mathcal{X}} \xrightarrow{\pi} \underset{\text{alg space}}{X}$ s.th.

(i) π strong homeomorphism ($|\mathcal{X}| \to |X|$ homeo, also after base change)
$\mathcal{X} \to \mathcal{X} \times_X \mathcal{X}$ univ. submersive
(e.g. separated univ homeo or flat univ homeo)

(ii) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$

<u>Fact</u>: π cms $\Rightarrow$ π initial among maps to alg. spaces.

<u>Fact</u>: A stack $\mathcal{X}$ w/ finite inertia $I_{\mathcal{X}} \to \mathcal{X}$ $\Rightarrow$ $\exists$ $\mathcal{X} \xrightarrow{\pi} X$ cms (and π separated)

<u>Ex</u>: G finite group scheme $\circlearrowright X$ qproj. $[X/G] \to X/G$ cms

If $\underset{\text{Spec } A}{\overset{\shortparallel}{X}}$ affine: $X/G = \operatorname{Spec} A^G$

<u>Ex</u>: $\mathcal{M}_g \to M_g$

§1.8 Good moduli spaces

[Alp13]

Def: A <u>good moduli space</u> is a map $\underset{\text{alg st.}}{\mathcal{X}} \xrightarrow{\pi} \underset{\text{alg sp}}{X}$ s.th

(0) π qcqs.

(i) π cohomologically affine: $\pi_*: QCoh(\mathcal{X}) \to QCoh(X)$ exact (also after b.c.)

(ii) $\pi_* \mathcal{O}_{\mathcal{X}} = \mathcal{O}_X$

Fact: If π gms: [Alp13] 4.5, 4.7, 4.16

(a) π initial ~~among~~ among maps to alg. spaces. (also [AHR2, Thm 3.11])

(b) π universally closed (but in general <u>not</u> homeo)

(c) $\forall x \in |X|$, $\exists!$ closed pt $x_0 \in \pi^{-1}(x)$ and ($\Rightarrow$ conn. fibers)

- $Aut(x_0)$ linearly reductive
- $\dim Aut(x_0) > \dim Aut(y)$ $\forall y \neq x_0 \in \pi^{-1}(x)$.

(d) π cms $\Leftrightarrow$ $\dim Aut(x)$ loc. constant.

(e) If
$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$
then
- π' gms
- $g^* \pi_* = \pi'_* g'^*$

(f) If ~~$\mathcal{X} = Spec_{\mathcal{X}} \mathcal{A}$~~, $\mathcal{A}$ qcoh $\mathcal{O}_{\mathcal{X}}$-alg, then
$$\begin{tikzcd} Spec_{\mathcal{X}} \mathcal{A} \arrow[r] \arrow[d, "gms"'] \arrow[dr, phantom, "\circ"] & \mathcal{X} \arrow[d, "\pi", "gms"'] \\ Spec_X \pi_* \mathcal{A} \arrow[r] & X \end{tikzcd}$$
(not cartesian)

(g) If $\mathcal{X}$ noetherian, then X noetherian and π_* preserves coherence.

(h) If $\mathcal{X}$ noetherian, $\Delta_{\mathcal{X}/X}$ affine, then π of finite type. [AHR1, Thm A.1]

(i) If $\mathcal{X} \xrightarrow{f.t.} \underset{noeth}{S}$ $\Rightarrow$ $X \to S$ f.t. ([Alp14] for S noeth, [Alp13] for S exc.)

(j) <u>Projection formula</u>: $\pi_*(\mathcal{F} \otimes \pi^* \mathcal{G}) \xleftarrow{\cong} \pi_* \mathcal{F} \otimes \mathcal{G}$

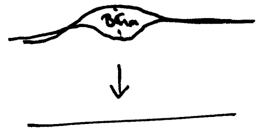
§1.9 Examples of good moduli spaces

G/k linearly reductive $\left(\begin{array}{l} \text{char } 0: \Leftrightarrow G^\circ \text{ reductive} \\ \text{char } p: \Leftrightarrow \underbrace{G^\circ \text{ diagonalizable}}_{\Updownarrow} + |G/G^\circ| \text{ prime to } p \end{array} \right.$

product ~~extension~~ of torus and ~~by~~ Cartier dual of abelian p-group   e.g. $\mathbb{G}_m^n \times \mu_p^3 \times \mu_{p^2}$

$G \circlearrowright \operatorname{Spec} A = X$

$$\begin{array}{c} [X/G] \\ \downarrow \text{gms} \\ X/\!\!/G = \operatorname{Spec} A^G \end{array}$$

Ex: $\begin{array}{c} [\mathbb{A}^1/\mathbb{G}_m] \\ \downarrow \\ * \end{array}$ $\begin{array}{c} [\mathbb{A}^2/\mathbb{G}_m] = B\mathbb{G}_m \quad \mathbb{P}^1 \\ \text{wts } 1,1 \\ \downarrow \\ * \end{array}$ $\begin{array}{c} [\mathbb{A}^2/\mathbb{G}_m] \\ \text{wts } 1,-1 \\ \downarrow \\ \mathbb{A}^1 \end{array}$ =  $= \operatorname{Spec} k[xy]$

Non-ex: $\mathbb{G}_m \circlearrowright \mathbb{P}^1$ $[\mathbb{P}^1/\mathbb{G}_m] = \underset{\substack{\| \\ 0 \\ \text{closed}}}{B\mathbb{G}_m} \quad \underset{\text{open}}{*} \quad \underset{\substack{\| \\ \infty \\ \text{closed}}}{B\mathbb{G}_m}$

$\lambda.(x:y) = (\lambda x : y)$

Any map $[\mathbb{P}^1/\mathbb{G}_m] \to X$ identifies the 3 pts $\Rightarrow$ not unique closed pt in fiber.

Every orbit must contain a unique closed orbit in its closure.

Ex: $GL_n \circlearrowright \mathfrak{gl}_n = \mathbb{A}^{n^2}$ acting by conjugation, over $k = \bar{k}$ char $= 0$.

$\mathcal{X} = [\mathfrak{gl}_n / GL_n]$ closed $\bar{k}$-points $\leftrightarrow$ diagonalizable matrices
$\bar{k}$-points $\leftrightarrow$ Jordan forms

$$\begin{bmatrix} \lambda_1 & 1 & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \overset{\text{spec.}}{\rightsquigarrow} \operatorname{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$$

$$\begin{array}{ccc} \mathbb{A}^{n^2} & \to & \mathcal{X} \\ & \searrow^{\chi} & \downarrow \pi \text{ gms} \\ & & \mathbb{A}^n \end{array}$$

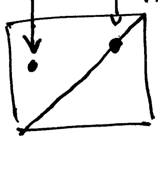
$\chi(A)$ char. pol. $= \{e_1(A), e_2(A), \dots, e_n(A)\}$

Ex: $n = 2$

$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ open pt in fiber

$\mathcal{X} \ni \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$ closed pt in fiber X_0

$\downarrow$

$\mathbb{A}^2 =$  $\lambda_1 = \lambda_2 \quad (e_1^2 = 4e_2)$

$\operatorname{stab}\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = GL_2$ $\operatorname{stab}\left(\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\right) = \mathbb{G}_m \times \mathbb{G}_a$

$\operatorname{stab}\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}\right) = \mathbb{G}_m^2$

§1.10 GIT and good moduli spaces

- G linearly reductive $\circlearrowright X$ projective $/k$
- $\mathcal{L}$ G-linearized ample line bundle

(roughly equiv to $X \xhookrightarrow{|\mathcal{L}|} \mathbb{P}(V)$ V linear rep. of G.)

$$\begin{tikzcd}
\overset{\text{stable locus}}{X^s(\mathcal{L})} \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d] & \overset{\text{semi-stable locus}}{X^{ss}(\mathcal{L})} \arrow[r, phantom, "\overset{\text{open}}{\subset}"] \arrow[d] & X \arrow[d, "\pi"] & \mathcal{L} = \pi^* \underline{\mathcal{L}} \\
{[X^s(\mathcal{L})/G]} \arrow[r, phantom, "\subset"] \arrow[d, "\text{cms}"] & {[X^{ss}(\mathcal{L})/G]} \arrow[r, phantom, "\subset"] \arrow[d, "\text{gms}"] & {[X/G]} & \underline{\mathcal{L}} \\
X^s(\mathcal{L})/\!/G \arrow[r, phantom, "\subset"] & X^{ss}(\mathcal{L})/\!/G & &
\end{tikzcd}$$

$$X^{ss}(\mathcal{L})/\!/G = \operatorname{Proj}\left(\bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^{\otimes n})^G\right) = \operatorname{Proj}\left(\bigoplus \Gamma(\mathcal{X}, \underline{\mathcal{L}}^{\otimes n})\right)$$

<u>Def</u>: $X^{ss}(\mathcal{L}) = \{x \in X : \exists n \geq 0,\ s \in \Gamma(X, \mathcal{L}^{\otimes n})^G,\ s(x) \neq 0\}$

exactly the locus where $\mathcal{X} = [X/G] \overset{\underline{\mathcal{L}}^{\otimes n}}{\dashrightarrow} \mathbb{P}(\Gamma(\mathcal{X}, \underline{\mathcal{L}}^{\otimes n}))$

defined for suff. divisible n.

<u>Excs</u> 1.4: (a) G lin red $\circlearrowright X = \operatorname{Spec} A$ $\Rightarrow$ $[X/G] \xrightarrow[\text{gms}]{\pi} X/\!/G = \operatorname{Spec} A$

(b) G lin red $\circlearrowright X$ proj, $\mathcal{L}$ $\Rightarrow$ $[X^{ss}(\mathcal{L})/G] \xrightarrow[\text{gms}]{} X^{ss}(\mathcal{L})/\!/G = \operatorname{Proj}(--)$

(c) G lin red $\circlearrowright X$ qproj, $\mathcal{L}$ $\quad X^{ss}(\mathcal{L}) = \{x \in X : \ \cdots \ + X_s \text{ affine}\}$

Prove $[X^{ss}(\mathcal{L})/G] \xrightarrow{\text{gms}} U \subset \operatorname{Proj}(\bigoplus \Gamma(X, \mathcal{L}^{\otimes n})^G)$ for suitable open U.

§1.11 Example of GIT

Ex: (4 ordered points on $\mathbb{P}^1$)

$$PGL_2 \circlearrowright X = (\mathbb{P}^1)^4$$

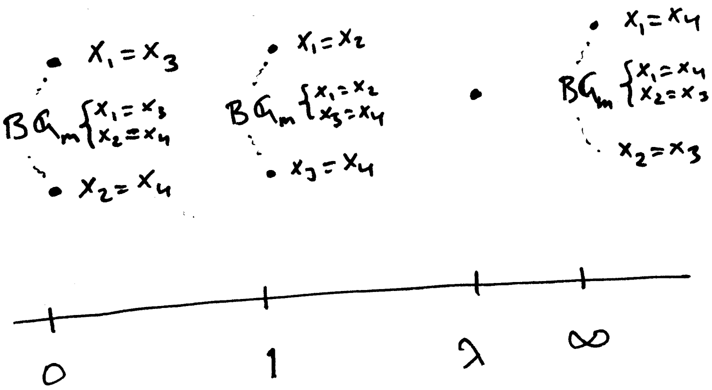
cross-ratio $\quad (\mathbb{P}^1)^4 \overset{\pi}{\dashrightarrow} \mathbb{P}^1$

$$(x_1, x_2, x_3, x_4) \longmapsto (x_1, x_2 ; x_3, x_4) := \frac{x_3 - x_1}{x_3 - x_2} \cdot \frac{x_4 - x_2}{x_4 - x_1}$$

π well-defined on $\quad X^{ss}$ = no three pts coincide

$\qquad\qquad\qquad\qquad\quad X^{s}$ = no two pts coincide

$$\begin{tikzcd}
X^s \arrow[d] & \subset & X^{ss} \arrow[d] & \subset X \\
{[X^s/PGL_2]} \arrow[d, equal] & \subset & {[X^{ss}/PGL_2]} \arrow[d] & = \\
X^s/PGL_2 \arrow[d, equal] & \subset & X^{ss}/\!/PGL_2 \arrow[d, equal] & = \\
\mathbb{P}^1 \setminus \{0,1,\infty\} & \subset & \mathbb{P}^1 &
\end{tikzcd}$$



Fibers over $0, 1, \infty$: $\quad [\mathbb{A}^1/\mathbb{G}_m] \coprod_{B\mathbb{G}_m} [\mathbb{A}^1/\mathbb{G}_m]$.

Fiber over $\lambda \notin \{0,1,\infty\}$: single pt w/o stab $\longleftrightarrow$ orbit of $(\infty, 0, 1, \lambda)$