LOCAL STRUCTURE OF ALGEBRAIC STACKS: EXERCISES
LECTURE 1: QUOTIENTS STACKS AND GOOD MODULI SPACES

* = more difficult

Exercise 1.1. Exercise on groupoids:

(a) Show that $\mathrm{Hom}_{\mathbf{Grpd}}(BG, BH) = [\mathrm{Hom}_{\mathrm{Grp}}(G, H)/H]$ where H acts by conjugation.

(b) Give a description of the 2-fiber product in **Grpd** and its universal property. Check the examples $G = * \times_{BG} *$ and $\mathrm{Ar}(\mathscr{X}) = \mathrm{Ob}(\mathscr{X}) \times_{\mathscr{X}} \mathrm{Ob}(\mathscr{X})$.

(c) Let $1 \to H \to G \to G/H \to 1$ be an exact sequence of groups. There are natural maps:

$$\begin{array}{ccccc}
H & \longrightarrow & * & & \\
\downarrow & & \downarrow & & \\
G & \longrightarrow & G/H & \longrightarrow & * \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & BH & \longrightarrow & BG \\
& & \downarrow & & \downarrow \\
& & * & \longrightarrow & B(G/H).
\end{array}$$

Show that every square is 2-cartesian.

Exercise 1.2 (*). In Set every G-torsor is trivial but in **Grpd** there are non-trivial G-torsors. The bottom square above exhibits $BH \to BG$ as a G/H-torsor. Note that $BH \to BG$ has no section (unless $H = G$). In general, if $X \to Y$ is a G-torsor, then $Y = [X/G]$. Show that G/H acts on the groupoid BH such that the quotient is BG (make these notions precise!). Also, when the sequence is split, show that the action is induced by an action of G/H on H.

Exercise 1.3 (*). For the Kummer exact sequence $1 \to B\boldsymbol{\mu}_n \to B\mathbb{G}_m \to B\mathbb{G}_m \to 1$, describe the action of $\mathbb{G}_m$ on $B\boldsymbol{\mu}_n$. Note that the sequence is not split.

Exercise 1.4. Let G be a linearly reductive group scheme over a field k.

(a) Show that $[\operatorname{Spec} A/G] \to \operatorname{Spec} A/\!\!/G := \operatorname{Spec} A^G$ is a good moduli space.

(b) If G acts on a projective scheme X equipped with a G-linearized ample line bundle, show that

$$[X^{ss}(\mathcal{L})/G] \to X^{ss}(\mathcal{L})/\!\!/G := \operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n)^G$$

is a good moduli space.

(c) When X is quasi-projective, the correct definition of $X^{ss}(\mathcal{L})$ is:

$$X^{ss}(\mathcal{L}) := \{x \in |X| \ : \ \exists n \geq 0, s \in \Gamma(X, \mathcal{L}^n)^G \ : \ s(x) \neq 0, X_s \text{ is affine}\}$$

Show that $[X^{ss}(\mathcal{L})/G]$ has a good moduli space which is an open subscheme of $\operatorname{Proj} \bigoplus_{n\geq 0} \Gamma(X, \mathcal{L}^n)^G$.

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