DAG-seminar #9

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I Def of tri. cat.

II Triangulated struct on $D(\mathcal{A})$

III Tri. cat w/ duality

IV Witt grps of tri cat w/ duality

I Def

$\mathcal{C}$ additive category.

A <u>triangulation</u> on $\mathcal{C}$ is:

- $\Sigma : \mathcal{C} \to \mathcal{C}$ additive automorphism <u>translation functor</u> $\quad (X[n] := \Sigma^n X)$
- A collection of distinguished triangles

$$X \to Y \to Z \to X[1]$$

satisfying:

<u>TR1</u>: $X = X \to 0 \to X[1]$ is dist.

given $X \xrightarrow{f} Y \quad \exists \; X \to Y \to Z \to X[0]$ dist

(Z unique up to non-unique iso)

triangles are closed under iso

<u>TR2</u>: given $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ dist

$$\left.\begin{array}{l} \Rightarrow Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1] \\ \text{and } Z[-1] \to X \to Y \to Z \end{array}\right\} \text{dist}$$

<u>TR3</u>:

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dotted, "\exists"'] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

<u>TR4</u>: Given

$$\alpha : X \xrightarrow{u} Y \to Z' \to$$
$$\beta : Y \xrightarrow{v} Z \to X' \to \qquad \triangle\text{'s}$$
$$\gamma : X \xrightarrow{vu} Z \to Y' \to$$

$$\exists \; \delta : Z' \to Y' \to X' \xrightarrow{\rho} \qquad \triangle$$

s.th octahedron property holds.

II Triangulated cat $D(\mathcal{A})$

Let $\mathcal{A}$ abelian category, $Ch(\mathcal{A})$ cat. of cochain cplx.

<u>Def</u>: <u>Shift functor</u>

$$\Sigma : Ch(\mathcal{A}) \to Ch(\mathcal{A})$$
$$(A, d) \mapsto A[1]$$

$$A[k] = \left(A^{n+k}, (-1)^k d_A^{n+k} \right)$$

Mapping cone of $f : A \to B$:

$$c(f) = \left(B^n \oplus A^{n+1}, \begin{pmatrix} d_B^n & f^{n+1} \\ 0 & -d_A^{n+1} \end{pmatrix} \right)$$

<u>Recall</u>. $K(\mathcal{A})$

<u>Fact</u>: The shift functor Σ and the triangles iso to triangles of the form

$$A \xrightarrow{f} B \to C(f) \to A[1]$$

$$B^n \to B^n \oplus A^{n+1} \to A^{n+1}$$

gives a triangulated structure on $K(\mathcal{A})$.

<u>Def</u>: A sequence $A^\bullet \to B^\bullet \to C^\bullet$ is <u>exact</u> (resp. <u>split</u>) if it is so degreewise.

<u>Lemma 1</u>: Let $\mathcal{E}$: $A^\bullet \xrightarrow{f} B^\bullet \to (B/A)^\bullet$ be exact, then the net maps φ

$$\begin{tikzcd} A^n \arrow[r] & B^n \arrow[r] & B^n/A^n \\ A^n \arrow[r] \arrow[u, equal] & B^n \arrow[r] \arrow[u, equal] & C(f) = B^n \oplus A^{n+1} \arrow[u, "\varphi^n"', "(pr, \sigma)"] \end{tikzcd}$$

form a chain map which is always a q-iso and φ homotopy equiv if f is split.

To check TR1:

- closed under isos (by def)
- $A^\bullet \xrightarrow{f} B^\bullet \to C(f)^\bullet \to A^\bullet[1]$
- $$\begin{tikzcd} A^\bullet \arrow[r, "id"] & A^\bullet \arrow[r] & 0 \\ A^\bullet \arrow[r] \arrow[u, equal] & A^\bullet \arrow[r] \arrow[u, equal] & C(id) \arrow[u, "s"] \end{tikzcd}$$

<u>Def</u>: Let $\mathcal{C}$ be a tri. cat, W a class of maps in $\mathcal{C}$.
A <u>Verdier localization</u> is an exact functor (i.e. pres. triangles and shift)

$$\gamma : \mathcal{C} \longrightarrow \mathcal{C}[W^{-1}]^\Delta$$

taking W to isos. and ~~with~~ univ. prop:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{D} \\ \mathcal{C}[W^{-1}] \arrow[ur, dashed, "\exists! \bar{F}"'] & \end{tikzcd}$$

<u>Fact</u>: The localization $K(\mathcal{A}) \longrightarrow D(\mathcal{A})$
is a Verdier localization where the dist. triangles in $D(\mathcal{A})$
are those triangles iso to (the image of) a dist triangle in $K(\mathcal{A})$.

$A^\bullet \xrightarrow{f^\bullet} B^\bullet \longrightarrow (B/A)^\bullet$ exact seq in $Ch(\mathcal{A})$ is q-iso to

$A^\bullet \longrightarrow B^\bullet \longrightarrow C(f)^\bullet$ which is dist. in $D(\mathcal{A})$.

III Triangulated categories w/ duality

<u>Def</u>: Let $\mathcal{C}$ be a triangulated category. A duality on $\mathcal{C}$ is a triple (D, δ, ϖ) s.th.

- $\delta = \pm 1$
- $D: \mathcal{C} \longrightarrow \mathcal{C}$ contravariant and δ-exact i.e.
  - * $\Sigma D = D \Sigma^{-1}$
  - * $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ dist

    $\Rightarrow D(C) \xrightarrow{D(v)} D(B) \xrightarrow{D(u)} D(A) \xrightarrow{\delta D(w)[1]} (D(C))[1]$

- ϖ is a nat iso $\mathbb{1} \Rightarrow D^2$ s.th.* $\forall A \in \mathcal{C}$

$$\begin{tikzcd} DA & D^3A \arrow[l, "D(\varpi_A)"'] \\ DA \arrow[u, equal] \arrow[ur, "\varpi_{DA}"'] \arrow[u, phantom, shift left=4, "\circ"] & \end{tikzcd}$$

  - * $\Sigma(\varpi_A) = \varpi_{\Sigma A}$

Example: Let $(\mathcal{C}, D, \delta, \varpi)$ be a triangulated cat w/ duality and $i \in \mathbb{Z}$.

Define

- $D^{[i]} = \Sigma^i D$
- $\delta^{[i]} = (-1)^i \delta$
- $\varpi^{[i]} = \delta^i (-1)^{\frac{i(i+1)}{2}} \varpi$

Then $(D^{[i]}, \delta^{[i]}, \varpi^{[i]})$ duality on $\mathcal{C}$ called i-shifted duality.

Let's check that $D^{[i]}$ is $\delta^{[i]}$-exact:

Given $A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} A[1]$ dist triangle we have

$$D(C) \xrightarrow{D(v)} D(B) \xrightarrow{D(u)} D(A) \xrightarrow{\delta\Sigma(Dw)} \Sigma D(C)$$

Hence

$$\Sigma^i D(C) \xrightarrow{(-1)^i \Sigma^i D(v)} \Sigma^i D(B) \xrightarrow{(-1)^i \Sigma^i D(u)} \Sigma^i D(A) \xrightarrow{(-1)^i \delta \Sigma^{i+1} D(w)} \Sigma^{i+1} D(C) \quad \text{dist}$$

$$\Rightarrow \quad \text{—}''\text{—} \xrightarrow{\Sigma^i D(v)} \text{—}''\text{—} \xrightarrow{\Sigma^i D(u)} \text{—}''\text{—} \xrightarrow{(-1)^i \delta \Sigma^{i+1} D(w)} \text{—}''\text{—} \quad \text{dist}$$

Side note: In general If $X \to Y \to Z \to X[1]$ dist, then (TR1)

$$Y \to Z \to X[1] \xrightarrow{-1} Y[1] \text{ dist}$$

$$\vdots$$

$$X[1] \xrightarrow{-1} Y[1] \xrightarrow{-1} Z[1] \xrightarrow{-1} X[1]$$

and

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "g"] \arrow[d, "-1"] & Z \arrow[r, "h"] \arrow[d, equal] & X[1] \\
X \arrow[r, "-f"] & Y \arrow[r, "-g"] & Z \arrow[r, "h"] & X[1]
\end{tikzcd}$$

<u>Example</u>: X regular scheme, $\mathcal{P}(X)$ loc free coherent $\mathcal{O}_X$-modules

If $\mathcal{F}^\bullet \in D^b(\mathcal{P}(X))$ bounded cplx of loc free coh $\mathcal{O}_X$, then

$\mathcal{F}^{\bullet\vee} := (\mathcal{F}^{n\vee})$ gives duality on $D^b(\mathrm{Coh}X)$

<u>Def</u>: A map $\varphi: A \to D^{(i)}A$ is i-symmetric if

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d, "\omega_A^{[i]}"'] & D^{(i)}A \arrow[d, equal] \\ (D^{[i]})^2 A \arrow[r, "D^{[i]}\varphi"'] & D^{[i]}A \end{tikzcd}$$

Let $\mathrm{Sym}^i(\mathcal{C})$ be the monoid of isometry classes of i-symm pairs

w/ addition given by the orthogonal sum.

<u>Rmk</u>: $\mathrm{Sym}^i_{D,\delta,\omega}(\mathcal{C}) = \mathrm{Sym}^0(\mathcal{C}, D^{[i]}, \delta^{[i]}, \omega^{[i]})$

<u>Def</u>: (A,φ) and (B,ψ) are <u>isometric</u> if $\exists f$

$$\begin{tikzcd} A \arrow[r, "f", "\cong"'] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & B \arrow[d, "\psi"] \\ D^{[i]}A & D^{[i]}B \arrow[l, "D^{[i]}f"] \end{tikzcd}$$

The <u>orthogonal sum</u> is $(A,\varphi) \oplus (B,\psi) = (A \oplus B, D^{[i]}(\varphi + \psi))$

$= (A \oplus B, D^{[i]}\varphi + D^{[i]}\psi)$

(Balmer ?)

Thm: Suppose $\mathcal{C}$ uniquely 2-divisible (i.e. all Hom-sets 2-divisible)

Let $A \xrightarrow{\varphi} D^{[i]}A$ be an i-symm pair. Then $\exists$ dist triangle

$$\begin{tikzcd} A \arrow[r, "\varphi"] & D^{[i]}A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circ"] & C \arrow[r, "\beta"] \arrow[d, dashed, "\exists \psi", "\wr"'] & A[1] \\ \text{plus} & D^{[i]}A \arrow[r, "\delta^{[i+1]} D^{(i+1)}\beta"'] & D^{[i+1]}C & \end{tikzcd}$$

s.th. (C, ψ) is $(i+1)$-symmetric and moreover (C,ψ) unique up to isometry

We get a well-defined map:

$$\mathrm{Symm}^i(\mathcal{C}) \xrightarrow{d^i} \mathrm{Symm}^{i+1}(\mathcal{C})$$
$$(A, \varphi) \longmapsto (C, \psi)$$

Moreover $d^{i+1}d^i = 0$ and d^i respects $\oplus$.

Def: The Witt group of $\mathcal{C}$:

$$W^i = H^i(\mathrm{Symm}^\bullet(\mathcal{C}), d^\bullet) = \ker(d^i)/\mathrm{im}(d^{i-1})$$ (recall, they're monoids)

Rmk: This is a group. If $(A, \varphi) \in \ker d^i$, then

$$A \xrightarrow[\cong]{\varphi} D^{[i]}A \to 0$$

and (A,φ) has inverse $(A, -\varphi)$.