DAG seminar #8

T. Bauer 6/4-2018

Recall last time

k comm ring, A k-alg, M an A-module

$\Omega_{A/k}$ A-module of Kähler differentials

$\mathcal{L}_{A/k} := \Omega_{P_\bullet/k} \otimes_{P_\bullet} A$

where $k \rightarrowtail P_\bullet \xrightarrow{\sim} A$ a factorization into a cofib + triv fib.

III Properties

Thm (Flat base change)

Let $k \to k'$ flat homo. of rings, A k-alg, $B = A \otimes_k k'$.

Then $B \otimes_A \mathcal{L}_{A/k} \xrightarrow{\sim} \mathcal{L}_{B/k'}$.

pf: Let $P_\bullet \xrightarrow{\sim} A$ cofibrant resolution. Then $k' \otimes_k P$ is cofibrant $/k'$

and by flatness $k' \otimes_k P \to k' \otimes_k A = B$ is a weak equiv.

So it suffices to show that $B \otimes_A \Omega_{A/k} \cong \Omega_{B/k'}$. This holds w/o flatness:

$$\mathrm{Hom}_B(B \otimes_A \Omega_{A/k}, -) = \mathrm{Hom}_A(\Omega_{A/k}, -) = \mathrm{Der}_k(A, -)$$

$$= \mathrm{Der}_{k'}(k' \otimes_k A, -) = \mathrm{Hom}_B(\Omega_{B/k'}, -)$$

on B-modules.

Prop: If $S \subseteq k$ is a mult subset, $k \to R = S^{-1}k$. Then $\mathbb{L}_{R/k} \simeq 0$.

pf: $R \otimes_k R = R$. Since $\mathbb{L}_{R/k}$ simplicial R-module, $\mathbb{L}_{R/k} \otimes_k R \cong \mathbb{L}_{R/k}$

By flat base change:

$$\mathbb{L}_{R/k} = \mathbb{L}_{R/k} \otimes_k R \simeq \mathbb{L}_{R \otimes_k R/R} \simeq \mathbb{L}_{R/R} \simeq 0 \qquad \square$$

Thm: Given $k \to A \to B$, the sequence

$$B \otimes_A \mathbb{L}_{A/k} \longrightarrow \mathbb{L}_{B/k} \longrightarrow \mathbb{L}_{B/A}$$

is a cofibration sequence.

pf: Choose a free resolution $P_\bullet \xrightarrow{\sim} A$. Pick a factorization:

$$\begin{tikzcd} P_\bullet \arrow[r, two heads, "\sim"] \arrow[d, tail] & A \arrow[d] \\ Q_\bullet \arrow[r, two heads, "\sim"] & B \end{tikzcd}$$

with $P_\bullet \rightarrowtail Q_\bullet$ free (e.g. bar resolution of $P_\bullet \to B$)

Get ~~right~~ level-wise exact seq (level-wise Ω):

$$0 \to \Omega_{P_\bullet/k} \otimes_{P_\bullet} Q_\bullet \xrightarrow[\text{inj}]{\text{split}^*} \Omega_{Q_\bullet/k} \longrightarrow \Omega_{Q_\bullet/P_\bullet} \longrightarrow 0$$

*b/c $P_\bullet \rightarrowtail Q_\bullet$ has levelwise retraction

Tensor $\otimes_{Q_\bullet} B$:

$$0 \to \Omega_{P_\bullet/k} \otimes_{P_\bullet} B \longrightarrow \Omega_{Q_\bullet/k} \otimes_{Q_\bullet} B \longrightarrow \Omega_{Q_\bullet/P_\bullet} \otimes_{Q_\bullet} B \longrightarrow 0$$

Take pushout of $P_\bullet \twoheadrightarrow A$ and $P_\bullet \rightarrowtail Q_\bullet$:

$$\begin{tikzcd} P_\bullet \arrow[r, two heads, "\sim"] \arrow[d, tail] & A \arrow[d, tail] \arrow[ddr, bend left] & \\ Q_\bullet \arrow[r, two heads, "\sim"] \arrow[drr, "\sim"'] & A \otimes_{P_\bullet} Q_\bullet \arrow[dr, "\text{w.eq by 2-out-of-3}"] & \\ & & B \end{tikzcd}$$

Thus $\mathbb{L}_{B/A} = \Omega_{A \otimes_{P_\bullet} Q_\bullet / A} \otimes_{A \otimes_{P_\bullet} Q_\bullet} B \overset{\text{b.c.}}{\cong} \left(\Omega_{Q_\bullet/P_\bullet} \otimes_{Q_\bullet} A \otimes_{P_\bullet} Q_\bullet\right) \otimes_{A \otimes_{P_\bullet} Q_\bullet} B = \Omega_{Q_\bullet/P_\bullet} \otimes_{Q_\bullet} B$

so get

$$0 \to \mathbb{L}_{A/k} \otimes_A B \to \mathbb{L}_{B/k} \to \mathbb{L}_{B/A} \to 0$$

<u>Cor</u>: $k \to R$, $S \subseteq R$ mult. Then

$$\mathbb{L}_{S^{-1}R/k} \simeq S^{-1}\mathbb{L}_{R/k}$$

<u>pf</u>: $k \to R \to S^{-1}R$ gives:

$$\begin{tikzcd} S^{-1}R \otimes_R \mathbb{L}_{R/k} \arrow[r] & \mathbb{L}_{S^{-1}R/k} \arrow[r] & \mathbb{L}_{S^{-1}R/R} \arrow[d, phantom, "\cong" rotate=90] \\ & & 0 \end{tikzcd}$$

$\square$

Let $r \in k$ and consider:

$$\begin{tikzcd} k \arrow[r] & k[x] \arrow[r] & k \\ & x \arrow[r, mapsto] & r \end{tikzcd}$$

$\Rightarrow$ cofiber seq:

$$\begin{tikzcd} k \otimes_{k[x]} \mathbb{L}_{k[x]/k} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{L}_{k/k} \arrow[r] \arrow[d, phantom, "="] & \mathbb{L}_{k/k[x]} \\ k\langle dx\rangle & 0 & \end{tikzcd}$$

so $\mathbb{L}_{k/k[x]} \simeq \cancel{\Sigma} k\langle dx\rangle[1]$ concentrated in deg 1.

Now consider

$$\begin{tikzcd} k[x] \arrow[r, two heads, "x \mapsto 0"] \arrow[d, "x \mapsto r"'] & k \arrow[d] \\ k \arrow[r, two heads] & k \otimes_{k[x]} k = k/(r) \end{tikzcd}$$

Take a resolution

$$\begin{tikzcd} k[x] \arrow[r, tail] \arrow[d] & B_\bullet \arrow[r, "\sim"] \arrow[d] & k \arrow[d] \\ k \arrow[r, tail] & A_\bullet \arrow[r] & k/(r) \end{tikzcd}$$

$A_\bullet = B_\bullet \otimes_{k[x]} k$

If $A_\bullet \xrightarrow{\sim} k/(r)$, then

$$\mathbb{L}_{(k/(r))/k} = \Omega_{A_\bullet/k} \otimes_{A_\bullet} k/(r) \simeq \Omega_{B_\bullet/k[x]} \otimes_{B_\bullet} A_\bullet \otimes_{A_\bullet} k/(r) =$$

$$= \Omega_{B_\bullet/k[x]} \otimes_{B_\bullet} k \otimes_k k/(r) = \mathbb{L}_{k/k[x]} \otimes_k k/(r) = k/(r)[1]$$

Check if $A_\bullet \xrightarrow{\sim} k/(r)$ on normalized chain complex.

Take $B_\bullet$ s.th. $NB_\bullet = k[x]\langle y\rangle \longrightarrow k[x]$
$y \longmapsto x$

$\Rightarrow NA_\bullet = k\langle y\rangle \longrightarrow k$
$y \longmapsto r$

This is a resolution of $k/(r)$ $\iff$ r is not a zero-divisor.

So $\mathbb{L}_{k/(r)/k} = k/(r)[1]$ if r not a zero-div.

Cor: If $k \longrightarrow R$ is locally a complete intersection then
flat dim$(\mathbb{L}_{R/k}) \leq 1$. (iff if k noeth and $k \to R$ f.t.)
Lichtenbaum – Schlessinger

IV Vistas

k noetherian, R f.g. k-alg.

$\mathbb{L}_{R/k} \simeq 0 \iff R/k$ étale

flat dim $\mathbb{L}_{R/k} = 0 \iff R/k$ smooth

Conj (Quillen) If $\mathbb{L}_{R/k}$ has finite flat dim, then flat dim $\mathbb{L} \leq 2$

proven: Avramov 2003: if $k \longrightarrow R$ has a section $s: R \longrightarrow k$
then f.dim $\mathbb{L}_{R/k} = 2 \iff s_I$ is an I-local complete intersection
$\Updownarrow$ for $I \supset \ker \ell$
f.dim $\mathbb{L}_{R/k} < \infty$