DAG-seminar #7

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The cotangent complex

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I Kähler differentials

Let k be a ring and A a k-algebra, M an A-module

Def: A *k-linear derivation* $A \xrightarrow{d} M$ is a k-linear map s.th.

$$d(ab) = ad(b) + d(a)b \qquad \text{(Leibniz rule)}$$

The set of such derivation is denoted $\mathrm{Der}_k(A,M)$

(k-linear $\Leftrightarrow$ $d|_k = 0$ under Leibniz rule)

The functor $M \mapsto \mathrm{Der}_k(A,M)$ is represented by an A-module $\Omega_{A/k}$, the module of *Kähler differentials*:

$$\mathrm{Der}_k(A,M) \cong \mathrm{Hom}_A(\Omega_{A/k}, M) \qquad \text{(natural in } M\text{)}$$

In particular, there's a universal derivation $A \xrightarrow{d} \Omega_{A/k}$.

Explicitly: $\Omega_{A/k} = A\langle da : a \in A\rangle / \sim$ where

- $d(a+b) \sim da + db$
- $d(ab) \sim adb + d(a)b$
- $d(\lambda) \sim 0$ if $\lambda \in k$

Square-zero extensions

$M \rightsquigarrow A \oplus M$ square-zero extension

mult: $(a,m)\cdot(a',m') = (aa', am' + a'm)$

$A \oplus M \twoheadrightarrow A$ surj of k-alg.

$(a,m) \longmapsto a$ w/ kernel M

$M^2 = 0$.

$$\mathrm{Der}_k(A,M) = \mathrm{Alg}_{k/A}(A, A\oplus M) = \{(f_0,f_1)\in \mathrm{Alg}_k(A, A\oplus M) : f_0 = \mathrm{id}_A\}$$

Ex: 1) $A = k[x_1,\dots,x_n] \Rightarrow \Omega_{A/k} = A\langle dx_1,\dots,dx_n\rangle$ free of rank n.

2) $A = k[x]/(x^n) \Rightarrow \Omega_{A/k} = A\langle dx\rangle / \langle nx^{n-1}dx\rangle$ not free (if $n\neq 0$ in k)

$0 = d(x^n) = nx^{n-1}dx$

Ex: If $k=\mathbb{R}$, $A = C^\infty(M)$, M smooth manifold, then $\mathrm{Der}_{\mathbb{R}}(A,\mathbb{R}) \cong T^d M \xleftarrow{d} C^\infty(M) = A$

"Def": If A/k flat of rel dim r, finitely presented and $\Omega_{A/k}$ is loc free of rank r, then A is smooth over k.

① Given $f: A\to B$ k-algebra map, we have

$$\mathrm{Der}_k(B,M) \xrightarrow{-\circ f} \mathrm{Der}_k(A,M)$$

and thus: $\Omega_{A/k} \xrightarrow{\Omega f/k} \Omega_{B/k}$ of A-modules.

$$\begin{tikzcd} \Omega_{A/k} \arrow[r] & \Omega_{B/k} \\ A \arrow[u, "d"] \arrow[r, "f"'] \arrow[ur, phantom, "\circ"] & B \arrow[u, "d"'] \end{tikzcd}$$

By extension of scalars, we have $B\otimes_A \Omega_{A/k} \to \Omega_{B/k}$.

② Given $g: k\to k'$, A k'-algebra, we have

$$\Omega_{A/k} \xrightarrow{\Omega A/g} \Omega_{A/k'} \quad \text{(since any } k'\text{-linear der is } k\text{-linear)}$$

Prop: Given any maps $k\to A\to B$, the sequence

$$(0\to)\ \Omega_{A/k}\otimes_A B \to \Omega_{B/k} \to \Omega_{B/A} \to 0$$

is right-exact, and also left-exact if B/A is (~~differentially~~ formally) smooth.

II The cotangent complex

$$\begin{tikzcd} sAlg_k \arrow[r, "forget"] & sMod_k \arrow[l, bend right, "Fr"'] \arrow[d, "N", "\wr\!\wr"'] \\ & Ch_k^{\geq 0} \end{tikzcd}$$ Dold-Kan

$sAlg_k$ = simplicial k-algebras

$A_2 \overset{d_0, d_1, d_2}{\rightrightarrows} A_1 \overset{d_0, d_1}{\rightrightarrows} A_0$, $s_0, s_1 : A_1 \to A_2$, $s_0 : A_0 \to A_1$

N normalization functor, e.g.,

$$N(M_\cdot)_n = M_n / s_i(M_{n-1})$$

$$d : N(M_\cdot)_n \longrightarrow N(M_\cdot)_{n-1}$$

Fr = free alg. functor = left adjoint to forget

We have a model structure on $Ch_k^{\geq 0}$ (proj model structure) where

w.e. = H_*-iso (quasi-iso)

fibrations = surjective in degrees > 0 (not 0!)

cofibrations = injective in all degrees w/ projective cokernels.

Correspondingly on $sMod_k$:

w.e. = π_*-equiv (homotopy eq.)

fibrations = componentwise surjective in degrees > 0

We obtain [Quillen] an induced model structure on $sAlg_k$ s.t.

$f : A_\cdot \to B_\cdot$ w.e. $\iff$ forget(f) w.e.
fib $\iff$ fib.

cofibrantly gen'd by Fr(cofibr.)

(works because has left adj and $sMod_k$ cofibr. gen'd)

<u>Def</u>: A <u>free extension</u> of simplicial algebras is ~~an alg map~~ a map of sAlg $A_\cdot \to B_\cdot$ s.th.

① $A_n \to B_n$ is $A_n \to A_n[S_n]$ polynomial ring for a set S_n

② $s_j(S_n) \subseteq S_{n+1}$ "degeneracy free"

<u>Prop</u>: Cofibrations in $sAlg_k$ are exactly the retracts of free extensions.

<u>Ex</u>: If $S_\bullet$ is a simplicial set, then $k[S_\bullet]$ is a cofibrant simplicial algebra.

<u>Def</u>: The <u>cotangent complex</u> of A/k is the simplicial A-module

$$\mathbb{L}_{A/k} := \Omega_{P_\bullet/k} \overset{\text{levelwise}}{\otimes_{P_\bullet}} A \qquad (\in sMod_k \text{ or } Ch_k^{\geq 0})$$

where $k \underset{\text{cofib}}{\rightarrowtail} P_\bullet \underset{\text{triv fib.}}{\xrightarrow{\sim}} A$ (a simplicial resolution)

Well-defined in homotopy category.

<u>Constructing resolutions</u>:

Bar construction:

$$\begin{tikzcd} & \vdots \\ & k[k[A]] \arrow[d, shift left=2, "d_1"] \arrow[d, shift right=2, "d_0"'] \\ k \arrow[r] & k[A] \arrow[r, "\varepsilon_A"'] \arrow[u, "s_0"'] & A \end{tikzcd} \qquad \mathrm{Bar}_k(A)$$

$$\begin{array}{c} [[a]] \\ \uparrow s_0 \\ [a] \xmapsto{\ \varepsilon\ } a \end{array} \qquad\qquad \begin{array}{l} d_0 = \varepsilon_{k[A]} \\ d_1 = k[\varepsilon_A] \end{array}$$

- $k \to \mathrm{Bar}_k(A)$ is <u>free</u>: $S_0 = A$, $S_1 = k[A]$, ...

  $s_j = \underbrace{k[\cdots k[}_{j} s_0] \cdots]$

- ε_A is a w.e.q: there's an extra degeneracy $s_{-1}: A \to k[A]$, $a \mapsto [a]$ <u>retraction of sSets</u>.

<u>Sample computation</u>

$\mathbb{L}_{A/\mathbb{Z}}$ where $A = \mathbb{Z}[x]/(x^n)$

Simplicial resolution:

start with $\mathbb{Z}[x] \twoheadrightarrow \mathbb{Z}[x]/(x^n)$

$\rightsquigarrow$ semisimplicial resolution

$$\underbrace{\begin{tikzcd} \mathbb{Z}[y] \arrow[d, shift right=2, "y \mapsto x^n"'] \arrow[d, shift left=2, "y \mapsto 0"] \\ \mathbb{Z}[x] \end{tikzcd}}_{\tilde{P}_\bullet} \xrightarrow{\ \sim\ } \mathbb{Z}[x]/x^n$$

Make it simplicial by a left Kan extension:

$$P_\bullet = \underset{\Delta^{op}_{inj} \to \Delta^{op}}{\mathrm{LKan}} \tilde{P}_\bullet \qquad (\Delta^{op}_{inj} \text{ semi-simplicial}$$

Explicitly: $P_\bullet \longrightarrow \mathbb{Z}[x]/(x^n)$

$$P_n = \mathbb{Z}[x, y_\sigma : \sigma : [n] \longrightarrow [1]]$$

$$(\Omega_{P_\bullet/k})_n = \mathbb{Z}[x, y_\sigma]\langle dx, dy_\sigma \rangle$$

$$\mathbb{L}_{A/k} = (\Omega_{P_\bullet/k}) \otimes_{P_\bullet} A \cong \mathbb{Z}[x]/(x^n) \langle dx, \underset{\deg 0}{dy_\sigma} \rangle \quad \deg |\sigma|$$

$$N\mathbb{L}_{A/k} = \begin{array}{c} A\langle dy \rangle \longrightarrow A\langle dx \rangle \\ dy \longmapsto n x^{n-1} dx \end{array}$$

$$\pi_0(\mathbb{L}_{A/k}) \cong \Omega_{A/k} = \mathbb{Z}[x]/(x^n) \langle dx \rangle / (n x^{n-1} dx)$$

$$\pi_1(\mathbb{L}_{A/k}) = H_1(N\mathbb{L}_{A/k}) = xA\langle dy \rangle / (x^n dy) \cong A/(x^{n-1})$$