

The cotangent complex

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I Kähler differentials

Let  $k$  be a ring and  $A$  a  $k$ -algebra,  $M$  an  $A$ -module

Def: A  $k$ -linear derivation  $A \xrightarrow{d} M$  is a  $k$ -linear map s.t.

$$d(ab) = ad(b) + d(a)b \quad (\text{Leibniz rule})$$

The set of such derivations is denoted  $\text{Der}_k(A, M)$

( $k$ -linear  $\Leftrightarrow d|_k = 0$  under Leibniz rule)

This functor  $M \mapsto \text{Der}_k(A, M)$  is represented by an  $A$ -module  $\Omega_{A/k}$ , the module of Kähler differentials:

$$\text{Der}_k(A, M) \cong \text{Hom}_A(\Omega_{A/k}, M) \quad (\text{natural in } M)$$

In particular, there's a universal derivation  $A \xrightarrow{d} \Omega_{A/k}$ .

Explicitly:  $\Omega_{A/k} = A \langle da : a \in A \rangle / \sim$  where

- $d(a+b) \sim da + db$
- $d(ab) \sim adb + d(a)b$
- $d(\lambda) \sim 0$  if  $\lambda \in k$

Square-zero extensions

$M \rightsquigarrow A \oplus M$  square-zero extension  
 with:  $(a, m) \cdot (a', m') = (aa', am' + a'm)$

$A \oplus M \twoheadrightarrow A$  surj of  $k$ -alg.  
 $(a, m) \mapsto a$  w/ kernel  $M$   
 $M^2 = 0$ .

$$\text{Der}_k(A, M) = \text{Alg}_{k/A}(A, A \otimes M) = \{ (f_0, f_1) \in \text{Alg}_k(A, A \otimes M) : f_0 = \text{id}_A \}$$

Ex: 1)  $A = k[x_1, \dots, x_n] \Rightarrow \Omega_{A/k} = A \langle dx_1, \dots, dx_n \rangle$  free of rank  $n$ .

2)  $A = k[x]/(x^n) \Rightarrow \Omega_{A/k} = A \langle dx \rangle / \langle nx^{n-1} dx \rangle$  not free (if  $n \neq 0$  in  $k$ )

$$0 = d(x^n) = nx^{n-1} dx$$

Ex: If  $k = \mathbb{R}$ ,  $A = C^\infty(M)$ ,  $M$  smooth manifold, then  $\text{Der}_{\mathbb{R}}(A, \mathbb{R}) \cong T^d M \xleftarrow{d} C^\infty(M) = A$

"Def": If  $A/k$  flat of rel dim  $r$ , finitely presented and  $\Omega_{A/k}$  is loc free of rank  $r$ , then  $A$  is smooth over  $k$ .

① Given  $f: A \rightarrow B$   $k$ -algebra map, we have

$$\text{Der}_k(B, M) \xrightarrow{- \circ f} \text{Der}_k(A, M)$$

and thus:  $\Omega_{A/k} \xrightarrow{\Omega f_k} \Omega_{B/k}$  of  $A$ -modules.

$$\begin{array}{ccc} d \uparrow & \circ & \uparrow d \\ A & \xrightarrow{f} & B \end{array}$$

By extension of scalars, we have  $B \otimes_A \Omega_{A/k} \rightarrow \Omega_{B/k}$ .

② Given  $g: k \rightarrow k'$ ,  $A$   $k'$ -algebra, we have

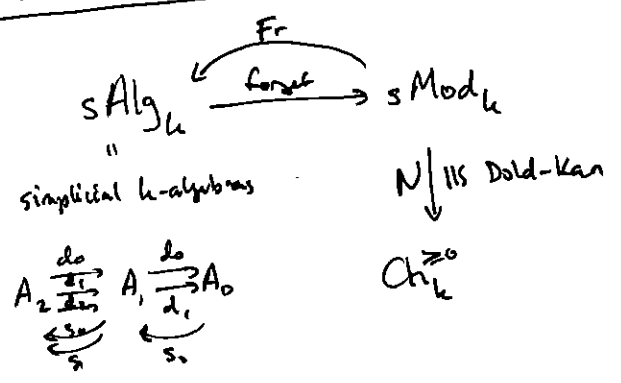
$$\Omega_{A/k} \xrightarrow{\Omega A/g} \Omega_{A/k'} \quad (\text{since any } k'\text{-linear der is } k\text{-linear})$$

Prop: Given any maps  $k \rightarrow A \rightarrow B$ , the sequence

$$(0 \rightarrow) \Omega_{A/k} \otimes_A B \rightarrow \Omega_{B/k} \rightarrow \Omega_{B/A} \rightarrow 0$$

is right-exact, and also left-exact if  $B/A$  is (differentially) smooth formally.

## II The cotangent complex



$N$  normalization functor, e.g.,

$$N(M_\bullet)_n = M_n / s_i(M_{n-1})$$

$$d: N(M_\bullet)_n \rightarrow N(M_\bullet)_{n-1}$$

$Fr =$  free alg. functor = left adjoint to forget

We have a model structure on  $Ch_k^{>=0}$  (proj model structure) where

w.e. =  $H_*$ -iso (quasi-iso)

fibrations = surjective in degrees  $> 0$  (not 0!)

cofibrations = injective in all degrees w/ projective cofibrants.

Correspondingly on  $sMod_k$ :

w.e. =  $\pi_*$ -equiv (homotopy eq.)

fibrations = componentwise surjective in degrees  $> 0$

We obtain [Quillen] an induced model structure on  $sAlg_k$  s.t.

$$f: A_\bullet \rightarrow B_\bullet \text{ w.e.} \iff \text{forget}(f) \text{ w.e.}$$

$$\text{fib} \iff \text{fib}$$

cofibrantly gen'd by  $Fr(\text{cofibr.})$

(works because has left cofib and  $sMod_k$  cofibr. gen'd)

Def: A free extension of simplicial algebras is ~~an alg map~~ <sup>a map of sAlg</sup>  $A_\bullet \rightarrow B_\bullet$  s.t.

①  $A_n \rightarrow B_n$  is  $A_n \rightarrow A_n[S_n]$  polynomial ring for a set  $S_n$

②  $s_j(S_n) \subseteq S_{n+1}$  "degeneracy free"

Prop: Cofibrations in  $sAlg_k$  are exactly the retracts of free extensions.

Ex: If  $S.$  is a simplicial set, then  $k[S.]$  is a cotangent simplicial algebra.

Def: The cotangent complex of  $A/k$  is the simplicial  $A$ -module

$$L_{A/k} := \Omega_{P./k} \otimes_{P.} A \quad (\text{ESMod}_k \text{ or } \mathcal{C}h_k^{zo})$$

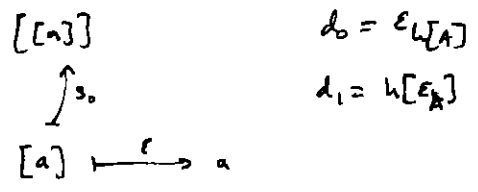
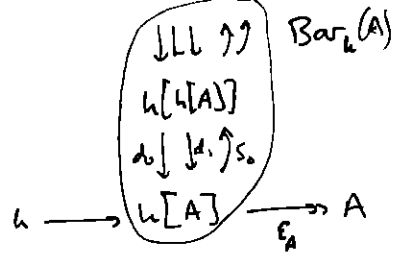
levelwise  
↓  
⊗  
P.

where  $k \rightarrow P. \xrightarrow{\sim} A$  (a simplicial resolution)

well-defined in homotopy category

Constructing resolutions:

Bar construction:



$k \rightarrow \text{Bar}_k(A)$   
 $\bullet$   $\mathbb{F}$  is free:  $S_0 = A, S_1 = k[A], \dots$

$$s_j = k[\dots k[s_0] \dots]$$

$\epsilon_A$   
 $\bullet$   $\mathbb{F}$  is a w.e.f.: there's an extra degeneracy  $s_{-1}: A \rightarrow k[A]$  retraction of sSets.  
 $a \mapsto [a]$

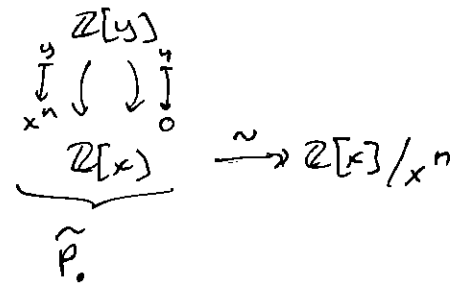
Sample computation

$$L_{A/\mathbb{Z}} \text{ where } A = \mathbb{Z}[x]/(x^n)$$

Simplicial resolution:

$$\text{start with } \mathbb{Z}[x] \twoheadrightarrow \mathbb{Z}[x]/(x^n)$$

$\rightarrow$  semisimplicial resolution



Make it simplicial by a left Kan extension:

$$P_\bullet = \text{LKan } \tilde{P}_\bullet \quad (\Delta_{inj}^{op} \text{ semi-simplicial})$$

$$\Delta_{inj}^{op} \rightarrow \Delta^{op}$$

Explicitly:  $P_\bullet \rightarrow \mathbb{Z}[x]/(x^n)$

$$P_n = \mathbb{Z}[x, y_\sigma : \sigma: [n] \rightarrow [1]]$$

$$(\Omega_{P_\bullet/k})_n = \mathbb{Z}[x, y_\sigma] \langle dx, dy_\sigma \rangle$$

$$\Omega_{A/k} = (\Omega_{P_\bullet/k}) \otimes_{P_\bullet} A \cong \mathbb{Z}[x]/(x^n) \langle dx, dy_\sigma \rangle$$

$$N_{A/k}^y = A \langle dy \rangle \rightarrow A \langle dx \rangle$$

$$dy \mapsto nx^{n-1} dx$$

$$\pi_0(N_{A/k}^y) \cong \Omega_{A/k} = \mathbb{Z}[x]/(x^n) \langle dx \rangle / (nx^{n-1} dx)$$

$$\pi_1(N_{A/k}^y) = H_1(N_{A/k}^y) = xA \langle dy \rangle / (x^{n-1} dy) \cong A/(x^{n-1})$$

