DAG-seminar #6

K. Moi 9/3-2018

Six functors in topology

1) Abelian sheaves on spaces

X top space

$Ab(X) :=$ category of sheaves of abelian groups on X

- presheaf: $F : Open(X)^{op} \longrightarrow Ab$
  + sheaf condition (unique gluing)
- natural transformations

(full subcategory of category of presheaves)

$Ab(X)$ is an abelian category. An exact sequence in $Ab(X)$

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

induces, for $U \subseteq X$ open, exact sequence

$$0 \longrightarrow F'(U) \longrightarrow F(U) \longrightarrow F''(U)$$

but $F(U) \longrightarrow F''(U)$ not necc surj:

$$F \longmapsto F(U) \quad \text{left exact but not necc right exact}$$

$Ab(X)$ has enough injectives.

Stalks: $x \in X$ point, $F \in Ab(X)$, stalk of F at $x \in X$: $\quad F_x := \operatorname*{colim}_{U \ni x} F(U)$

Fact: $0 \to F' \to F \to F'' \to 0$ exact $\iff$ $0 \to F'_x \to F_x \to F''_x \to 0$ exact $\forall x \in X$

In particular $F \longmapsto F_x$ is exact.

2) Operations

$f : X \longrightarrow Y$ continuous map

$f_* : Ab(X) \longrightarrow Ab(Y)$ direct image functor left exact but not necc right exact.

$$(f_* F)(V) = F(f^{-1}(V))$$

<u>f_* has a left adjoint</u>:

$$f^{-1} : Ab(Y) \longrightarrow Ab(X)$$

1) forget to presheaves: $Ab(Y) \longrightarrow Pre(Y)$
2) take left Kan extension
3) sheafify

$f^{-1}F$ is the sheafification of $V \longmapsto \operatorname{colim}_{U \supseteq f(V)} F(U)$

f^{-1} left adjoint $\Rightarrow$ preserves colimits $\Rightarrow$ right-exact

But also <u>left exact</u>!

If $x \in X$, w/ inclusion map $* \xhookrightarrow{i_x} X$, then $i_x^{-1} F = F_x$. (b/c sheafification preserves stalks)

so $(f^{-1}F)_x = i_x^{-1} f^{-1} F \cong i_{f(x)}^{-1} F = F_{f(x)} \Rightarrow f^{-1}$ exact.

$c_X : X \longrightarrow *$ "crushing X to a point"

<u>Def</u>: $c_X^{-1} A =: A_X$ constant sheaf. In particular $\mathbb{Z}_X$.

<u>Fact</u>: $\mathbb{Z}_X$ sheafification of $\mathbb{Z}\,Hom_{Open X}(-, X)$ (constant presheaf $\mathbb{Z}$!)

<u>Yoneda</u>: $F(X) \cong Hom_{PSh(X)}(\mathbb{Z} Hom(-,X), F) \cong Hom_{Ab(X)}(\mathbb{Z}_X, F)$

$\quad \Gamma(X, F) = F(X)$ $\qquad \cong Hom_{Ab}(\mathbb{Z}, (c_X)_* F) = (c_X)_* F$

so $\Gamma(X, -) = (c_X)_*$

<u>Skyscraper sheaf</u>:

$(i_x)_* A \in Ab(X)$ for $x \in X$, A abelian group.

2 more operations

Def: $F \otimes G$ = sheafification of $U \longmapsto F(U) \otimes G(U)$.

Def: $\mathcal{H}om(F, G)$ sheaf Hom:

$$\mathcal{H}om(F,G)(U) = \mathrm{Hom}_{Ab(U)}(F|_U, G|_U)$$

$- \otimes G$ left adjoint to $\mathrm{Hom}(G, -)$

Sheafified adjunction: $\mathcal{H}om_{Ab(X)}(F \otimes G, H) \cong \mathcal{H}om_{Ab(X)}(F, \mathcal{H}om(G, H))$

Exceptional direct image (or proper direct image, or direct image w/ compact support)

Now suppose X, Y are locally compact and Hausdorff (LCH)

$f: X \longrightarrow Y$.

Def: Given $s \in F(U)$, $\mathrm{supp}(s) = \{x \in U : s_x \neq 0\}$ $\quad \left(\begin{array}{ccc} F(U) & \longrightarrow & F_x \\ \rotatebox{90}{$\in$} & & \\ s & \longmapsto & s_x \end{array} \right)$

Def: $g: W \to Z$ proper (W, Z LCH) if g is closed with compact fibers.

(equiv: g^{-1}(compact) is compact)

Def: $f_! : Ab(X) \longrightarrow Ab(Y)$

$$(f_! F)(U) = \{ s \in F(f^{-1}(U)) : \ \mathrm{Supp}(s) \subseteq f^{-1}(U) \longrightarrow U \ \text{ is proper} \} \subseteq (f_* F)(U)$$

$\exists$ nat. trm: $f_! F \hookrightarrow f_* F$; an iso if f is proper.

Def: Sections w/ compact support: $\Gamma_c(X, F) = (\epsilon_X)_! F = \{ s \in F(X) : \mathrm{Supp}(s) \text{ compact} \}$

Derive:

Cochain complexes $F^\bullet = \to F^i \xrightarrow{d} F^{i+1} \to \dots$

$\mathcal{D}^*(X) := D^*(Ab(X))$

$$\begin{tikzcd} D^+(X) \arrow[r, bend left, "① Rf_*"] \arrow[r, phantom, "\top"] & D^+(Y) \arrow[l, bend left, "② f^{-1}"] \end{tikzcd}$$ adjunction (of triangulated functors)

Sheaf cohomology: $H^k(X,F) := R^k\Gamma(X,F) = H^k R(c_X)_* F$

Rmk: For nice spaces (locally contractible)

$$R\Gamma(X, \mathbb{Z}_X) \simeq C^*_{sing}(X, \mathbb{Z}) \quad \text{in } D(Ab)$$

$$\mathcal{H}om(-,-) \rightsquigarrow \overset{③}{R\mathcal{H}om}(-,-)$$ (injective res in 2^{nd} arg or proj res in 1^{st} arg)

For nice X also have ④ $- \overset{L}{\otimes} -$ (flat resolutions, for nice X stay in $D^+(X)$)

⑤ $Rf_!$

Exceptional inverse image $f^!$

If $f: X \to Y$ has fibers with finite $H^n(-, F)$ then

$\exists!\ f^!: D^+(Y) \to D^+(X)$ right adjoint to $f_!$

Def: $\omega_{X/Y} := f^! \mathbb{Z}_Y$ relative dualizing sheaf.

$\exists$ map $\omega_{X/Y} \overset{L}{\otimes} f^{-1}(F^\bullet) \xrightarrow{\Theta} f^!(F^\bullet)$

Fact: If $f: X \to Y$ fiber bundle w/ manifold fibers, then Θ is an iso.

Apply to $\underset{c_X}{\overset{\shortparallel}{f}}: X \to *$, $\underset{=M^n}{X}$ mfld. Then $\omega_M := \omega_{M/*}$ is $or_M[n]$

or_M is orientation sheaf: $or_{M,x} = H^n(M, M\setminus x, \mathbb{Z})$. In part, if M orientable: $\omega_M = \mathbb{Z}_M[n]$.

Poincaré duality

M manifold, dim n, orientable

$$R\Gamma_c(\cancel{M}, \mathbb{Q}_M)^\vee \cong R\Gamma(M, \mathbb{Q}_M)[n]$$

$$\overset{H^{-k}}{\Rightarrow} \quad H^k_c(M, \mathbb{Q})^\vee \cong H^{n-k}(M, \mathbb{Q})$$