DAG-seminar #5

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Derived categories & functors

$\mathcal{A}$ abelian category

$\mathrm{Kom}^*(\mathcal{A})$ = complexes in $\mathcal{A}$ $\qquad * \in \{+, -, b, \emptyset\}$

$K^*(\mathcal{A})$ = — " — modulo homotopies

$D^*(\mathcal{A})$ = derived category

Thm: $\exists Q : \mathrm{Kom}(\mathcal{A}) \longrightarrow D(\mathcal{A})$ sending quasi-iso to iso, which is universal w.r.t. this property.

Naive approach: Suppose $S \subset \mathrm{Mor}(\mathcal{C})$ class of morphisms in $\mathcal{C}$

$\mathcal{C}[S^{-1}]$: $\mathrm{Ob}\,\mathcal{C}[S^{-1}] = \mathrm{Ob}\,\mathcal{C}$

$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X,Y)$ = "paths $X \xrightarrow{f_1} Z_1 \xleftarrow{\in S} \xrightarrow{f_2} Z_2 \xleftarrow{\in S} \cdots \xleftarrow{\in S} Y$ " $/\sim$

Problem: not nec a set.

Solution: Localizing classes of morphisms.

Def: A class $S \subset \mathrm{Mor}\,\mathcal{C}$ is called localizing (or a multiplicative system) if:

(1) closed under composition (includes all identity morphisms)

(2) Ore condition:

$$\begin{tikzcd} W \arrow[r, dashed] \arrow[d, "S\ni"'] & Z \arrow[d, "s\in S"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} W & Z \arrow[l, dashed] \\ X \arrow[u, dashed, "S\ni"] & Y \arrow[l, "f'"] \arrow[u, "s'"'] \end{tikzcd}$$

(3) Cancellation law: $f,g : X \longrightarrow Y$. Then TFAE

(i) $\exists s : Z \longrightarrow X$ s.t. $fs = gs$ $\quad (s \in S)$

(ii) $\exists t : Y \longrightarrow W$ s.t. $tf = tg$ $\quad (t \in S)$

<u>Roof description</u> (prop)

If S localizing, then maps in $\mathcal{C}[S^{-1}]$ may be represented by left or right "roofs".

$$\begin{tikzcd} & Z \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} & Z & \\ X \arrow[ur, "g"] & & Y \arrow[ul, "t"'] \end{tikzcd}$$

and $(s, f) \sim (s', f')$ if $\exists$

$$\begin{tikzcd} & Z'' \arrow[dl, "s''"'] \arrow[dr, "f''"] & \\ Z \arrow[d, "s"'] \arrow[drr, "f"] & & Z' \arrow[dll, "s'"'] \arrow[d, "f'"] \\ X & & Y \end{tikzcd}$$

<u>Rmk</u>: Still no reason for $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ to be a set.

<u>Def</u>: A localizing class S is called locally small if $\forall X$, there is a set $S_X \subset S$ of morphisms w/ target X s.th.

$$\forall (X_1 \xrightarrow{s} X) \in S \quad \exists\, (X_2 \longrightarrow X_1 \xrightarrow{s} X) \in S_X$$

<u>Lemma</u>: If S locally small, then $\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$ is a set.

<u>pf</u>: Consider S_X as a small category: objects $= (\cancel{Z}_1 \xrightarrow{s} X) \in S_X$

morphism $= \left(\begin{tikzcd} Z_1 \arrow[rr] \arrow[dr, "s_1"'] & & Z_2 \arrow[dl, "s_2"] \\ & X & \end{tikzcd} \right)$

Ore condition $\Rightarrow$ can enlarge S_X to make it filtered. Then

$$\mathrm{Hom}_{\mathcal{C}[S^{-1}]}(X, Y) = \lim_{\substack{(X_1 \to X) \\ \in \\ S_X}} \mathrm{Hom}_{\mathcal{C}}(X_1, Y)$$

$$\begin{tikzcd} & X_1 \arrow[dl, "s \in S_X"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

<u>Thm</u>: The category $D(\mathcal{A})$ exists (within our universe) for every well-powered AB5-cat $\mathcal{A}$ which has a set of generators

<u>Thm</u>: $\mathcal{D}(\mathcal{A}) \cong K(\mathcal{A})[S^{-1}]$ where $S =$ quasi-iso.

<u>Ex</u>: If $\mathcal{A}$ has enough injectives, then $K^+(\mathcal{I}) \xrightarrow{\simeq} D^+(\mathcal{A})$

<u>Rmk</u>: We have strict implications:

$$\{f = 0 \text{ in } Kom(\mathcal{A})\} \Rightarrow \{f = 0 \text{ in } K(\mathcal{A})\}$$
$$\Rightarrow \{f = 0 \text{ in } D(\mathcal{A})\} \Rightarrow \{H^n(f) = 0 \ \forall n\}$$

<u>Ex</u>:
$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, ".2"] \arrow[d, "id"'] & \mathbb{Z} \arrow[r] \arrow[d, ".2"] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "quot"'] & \mathbb{Z}/3\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

<u>coh</u>:
$$\begin{array}{cccc} 0 & 0 & \mathbb{Z}/2\mathbb{Z} & 0 \\ 0 & \mathbb{Z} & 0 & 0 \end{array}$$

<u>Def</u>: $\mathcal{I} \subset \mathrm{Ob}\,\mathcal{A}$ injective objects

<u>Prop</u>: Let $\underset{K^+(\mathcal{I})}{\overset{\cap}{I^\bullet}} \xrightarrow{s} \underset{K^+(\mathcal{A})}{\overset{\cap}{K^\bullet}}$ be a quasi-iso. Then $\exists K^\bullet \xrightarrow{t} I^\bullet$ s.th. $ts \underset{\text{homotopic}}{\simeq} id_{I^\bullet}$

<u>Ex</u>: $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) := \mathrm{Hom}_{D(\mathcal{A})}(X[0], Y[i])$

(1) $\mathrm{Hom}_{K(\mathcal{A})}(X, I) \xrightarrow{\simeq} \mathrm{Hom}_{D(\mathcal{A})}(X, I)$

(2) $\mathrm{Ext}^i_{\mathcal{A}}(X, Y) = \mathrm{Hom}_{K(\mathcal{A})}(X, I[i]) = H^i(\mathrm{Hom}^\bullet(X, I))$

Derived functors

For left (or right) exact functor $F: \mathcal{A} \to \mathcal{B}$, want to construct

$$RF: D^+(\mathcal{A}) \to D^+(\mathcal{B}) \quad \left(\text{or } LF: D^-(\mathcal{A}) \to D^-(\mathcal{B})\right)$$

satisfying some universal property.

<u>Def</u>: A class of objs $R \subset \mathrm{Ob}\,\mathcal{A}$ is said to be <u>adapted</u> to a left exact functor F if:

a) F commutes w/ finite direct sums of objs in R

b) F sends acyclic complexes in $\mathrm{Kom}^+(R)$ to acyclic complexes.

c) Any object of $\mathcal{A}$ is a subobject of an object in R.

<u>Ex</u>: X scheme, $\mathcal{F}$ $\mathcal{O}_X$-module

(1) $F = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, -)$ then $R = \mathrm{Inj}\ \mathcal{O}_X\text{-mod}$ is adapted to F (left exact)

(2) $F = \mathcal{G} \otimes_{\mathcal{O}_X} -$ then $R = \text{flat } \mathcal{O}_X\text{-mod}$ is adapted to F (right exact)

<u>pf of (2)</u>: Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. For every open $j: U \subset X$ and every $s \in \mathcal{F}(U)$ we have a map $\mathcal{O}_U \to \mathcal{F}|_U$, $1 \mapsto s$

so by adjunction get $j_!\mathcal{O}_U \to \mathcal{F}$.

<u>Rmk</u>: $j_!\mathcal{O}_U$ is flat: $(j_!\mathcal{O}_U)_x = \begin{cases} 0 & \text{if } x \notin U \\ \mathcal{O}_{X,x} & \text{if } x \in U \end{cases}$

Take $\bigoplus_{(U,s)} j_!\mathcal{O}_U \twoheadrightarrow \mathcal{F}$. It's surjective. (check on stalks)

To prove b), enough to consider $0 \to F' \to F \to F''$ of flat $\mathcal{O}_X$-modules.

Write G as a quotient by a flat A:

$$\begin{array}{ccccccc}
0 \to & A\otimes F' & \to & A\otimes F & \to & A\otimes F'' & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& G\otimes F' & \to & G\otimes F & \to & G\otimes F'' & \to 0 \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

<u>Thm</u>: If $\mathcal{A}$ has enough injectives, then $\mathcal{I}$ adapted to <u>any</u> left exact functor F

<u>pf</u>: Let $I^\bullet = 0 \to I^0 \to I^1 \to \cdots$ acyclic

Enough to prove that $\mathrm{id}_{I^\bullet}$ is homotopic to zero (then $F(I^\bullet)$ also homotopic to zero)

More generally, take $C^\bullet = 0 \to C^0 \to C^1 \to \cdots$ acyclic, left-bounded

and $I^\bullet$ ~~acyclic~~ cplx of inj and $f: C^\bullet \to I^\bullet$ any map. Then $f \sim 0$.

<u>Thm</u>: Let R be a class adapted to some functor F. Let S_R be the class of q-iso in $K^+(R)$. Then

$$K^+(R)[S_R^{-1}] \longrightarrow D^+(\mathcal{A})$$

is an equivalence.

<u>Cor</u>: $K^+(\mathcal{I}) \cong D^+(\mathcal{A})$

<u>pf</u>: A quasi-iso in $K^+(\mathcal{I})$ is an iso.

<u>Def</u>: A right derived functor of a left exact functor F is a pair consisting of

$$RF: D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

together with

$$\begin{tikzcd} K^+(\mathcal{A}) \arrow[r, "K^+(F)"] \arrow[d, "Q_{\mathcal{A}}"'] \arrow[dr, phantom, "\Downarrow \varepsilon_F"] & K^+(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ D^+(\mathcal{A}) \arrow[r, "RF"] & D^+(\mathcal{B}) \end{tikzcd}$$