

Derived categories & functors

$\mathcal{A}$  abelian category

$Kom^*(\mathcal{A}) = \text{complexes in } \mathcal{A}$   $*$   $\in \{+, -, b, \emptyset\}$   
 $K^*(\mathcal{A}) = \text{---} \cup \text{---}$  modulo homotopies  
 $D^*(\mathcal{A}) = \text{derived category}$

Thm:  $\exists Q: Kom(\mathcal{A}) \rightarrow D(\mathcal{A})$  sending quasi-iso to iso, which is universal w.r.t. this property.

Naïve approach: Suppose  $S \subset Mor(\mathcal{C})$  class of morphisms in  $\mathcal{C}$

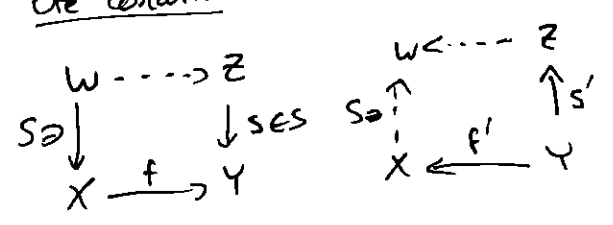
$\mathcal{C}[S^{-1}]$ :  $Ob \mathcal{C}[S^{-1}] = Ob \mathcal{C}$   
 $Hom_{\mathcal{C}[S^{-1}]}(X, Y) = \text{'paths' } X \xrightarrow{f_1} Z_1 \xleftarrow{res} Z_2 \xrightarrow{f_2} Z_3 \xleftarrow{res} \dots \xrightarrow{f_n} Y \text{ " / } \sim$

Problem:  $\nearrow$  not necc a set.

Solution: Localizing classes of morphisms.

Def: A class  $S \subset Mor \mathcal{C}$  is called localizing (or a multiplicative system) if:

- (1) closed under composition (includes all identity morphisms)
- (2) Ore condition:



(3) Cancellation law:  $f, g: X \rightarrow Y$ . Then TFAE

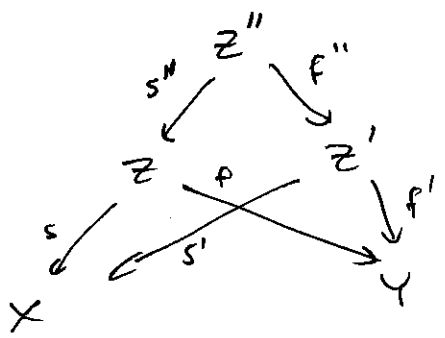
- (i)  $\exists s: Z \rightarrow X$  s.t.  $fs = gs$
- (ii)  $\exists t: Y \rightarrow W$  s.t.  $tf = tg$

Root description (prop)

If  $S$  localizing, then  $\text{map in } \mathcal{C}[S^{-1}]$  may be represented by left or right "roots".



and  $(s, f) \sim (s', f')$  if  $\exists$



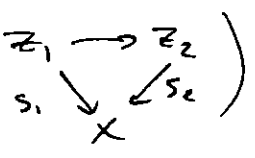
Rmb: Still no reason for  $\text{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$  to be a set.

Def: A localizing class  $S$  is called locally small if  $\forall X$ , there is a set  $S_X \subset S$  of morphisms w/ target  $X$  s.th.

$$\forall (X_1 \xrightarrow{s} X) \in S \quad \exists (X_2 \rightarrow X, \xrightarrow{s} X) \in S_X$$

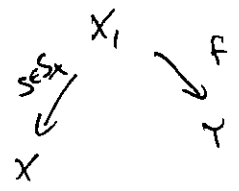
Lemma: If  $S$  locally small, then  $\text{Hom}_{\mathcal{C}[S^{-1}]}(X, Y)$  is a set.

pf: Consider  $S_X$  as a small category: objects =  $(Z_1 \xrightarrow{s} X) \in S_X$   
 morphism =  $(Z_1 \rightarrow Z_2)$



One condition  $\Rightarrow$  can enlarge  $S_X$  to make it filtered. Then

$$\text{Hom}_{\mathcal{C}[S^{-1}]}(X, Y) = \lim_{\substack{(X_i \rightarrow X) \\ \uparrow \\ S_X}} \text{Hom}_{\mathcal{C}}(X_i, Y)$$



Thm: The category  $D(\mathcal{A})$  exists (within our universe) for every well-powered ABS-cat  $\mathcal{A}$  which has a set of generators

Thm:  $D(\mathcal{A}) \cong K(\mathcal{A})[S^{-1}]$  where  $S = \text{quasi-iso}$ .

Ex: If  $\mathcal{A}$  has enough injectives, then  $K^+(\mathcal{I}) \xrightarrow{\cong} D^+(\mathcal{A})$

Rmk: We have strict implications:

$$\begin{aligned} \{f=0 \text{ in } \text{Kom}(\mathcal{A})\} &\Rightarrow \{f=0 \text{ in } K(\mathcal{A})\} \\ &\Rightarrow \{f=0 \text{ in } D(\mathcal{A})\} \Rightarrow \{H^n(f)=0 \forall n\} \end{aligned}$$

Ex:

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \rightarrow & 0 \\ & & \text{id} \downarrow & & \downarrow \cdot 2 & & \\ 0 & \rightarrow & \mathbb{Z} & \xrightarrow{\text{quot}} & \mathbb{Z}/2\mathbb{Z} & \rightarrow & 0 \end{array} \quad \begin{array}{ccccccc} \text{coh: } 0 & & 0 & & \mathbb{Z}/2\mathbb{Z} & & 0 \\ & & 0 & & \mathbb{Z} & & 0 \\ & & & & 0 & & 0 \end{array}$$

Def:  $\mathcal{I} \subset \text{Ob } \mathcal{A}$  injective objects

Prop: Let  $\mathcal{I}^\bullet \xrightarrow{s} \mathcal{K}^\bullet$  be a quasi-iso. Then  $\exists \mathcal{K}^\bullet \xrightarrow{t} \mathcal{I}^\bullet$  s.t.  $ts \cong \text{id}_{\mathcal{I}^\bullet}$  homotopic

Ex:  $\text{Ext}_{\mathcal{A}}^i(X, Y) := \text{Hom}_{D(\mathcal{A})}(X[0], Y[i])$

- (1)  $\text{Hom}_{K(\mathcal{A})}(X, \mathcal{I}) \xrightarrow{\cong} \text{Hom}_{D(\mathcal{A})}(X, \mathcal{I})$
- (2)  $\text{Ext}_{\mathcal{A}}^i(X, Y) = \text{Hom}_{K(\mathcal{A})}(X, \mathcal{I}[i]) = H^i(\text{Hom}^\bullet(X, \mathcal{I}))$

# Derived functors

For left (or right) exact functor  $F: \mathcal{A} \rightarrow \mathcal{B}$ , want to construct

$$RF: D^+(\mathcal{A}) \rightarrow D^+(\mathcal{B}) \quad (\text{or } LF: D^-(\mathcal{A}) \rightarrow D^-(\mathcal{B}))$$

satisfying some universal property.

Def: A class of objects  $\mathcal{R} \subset \text{Obj } \mathcal{A}$  is said to be adapted to a left exact functor  $F$  if:

- a)  $F$  commutes w/ finite direct sums of objects in  $\mathcal{R}$
- b)  $F$  sends acyclic complexes in  $\text{Kom}^+(\mathcal{R})$  to acyclic complexes.
- c) Any object of  $\mathcal{A}$  is a subobject of an object in  $\mathcal{R}$ .

Ex:  $X$  scheme,  $\mathcal{F}$   $\mathcal{O}_X$ -module

(1)  $F = \text{Hom}_{\mathcal{O}_X}(\mathcal{G}, -)$  then  $\mathcal{R} = \text{Inj } \mathcal{O}_X\text{-mod}$  is adapted to  $F$  (left exact)

(2)  $F = \mathcal{F} \otimes_{\mathcal{O}_X} -$  then  $\mathcal{R} = \text{Flat } \mathcal{O}_X\text{-mod}$  is adapted to  $F$  (right exact)

pf of (2): Let  $\mathcal{F}$  be an  $\mathcal{O}_X$ -module. For every open  $U \subset X$  and every  $s \in \mathcal{F}(U)$  we have a map  $\mathcal{O}_U \rightarrow \mathcal{F}(U)$   
 $1 \mapsto s$

so by adjunction get  $j_! \mathcal{O}_U \rightarrow \mathcal{F}$ .

Rem:  $j_! \mathcal{O}_U$  is flat:  $(j_! \mathcal{O}_U)_x = \begin{cases} 0 & \text{if } x \notin U \\ \mathcal{O}_{X,x} & \text{if } x \in U \end{cases}$

Take  $\bigoplus_{(U,s)} j_! \mathcal{O}_U \rightarrow \mathcal{F}$ . It's surjective. (check on stalks)

To prove b), enough to consider  $0 \rightarrow F' \rightarrow F \rightarrow F''$  of flat  $\mathcal{O}_X$ -modules.

Write  $G$  as a quotient by a flat  $A$ :

$$\begin{array}{ccccccc} 0 & \rightarrow & A \otimes F' & \rightarrow & A \otimes F & \rightarrow & A \otimes F'' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & G \otimes F' & \rightarrow & G \otimes F & \rightarrow & G \otimes F'' \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

Thm: If  $\mathcal{A}$  has enough injectives, then  $\mathcal{I}$  adapted to any left exact functor  $F$

ph: Let  $I^\bullet = 0 \rightarrow I^0 \rightarrow I^1 \rightarrow \dots$  acyclic

Enough to prove that  $\text{id}_{I^\bullet}$  is homotopic to zero (then  $F(I^\bullet)$  also homotopic to zero)

More generally, take  $C^\bullet = 0 \rightarrow C^0 \rightarrow C^1 \rightarrow \dots$  acyclic, left-bounded

and  $I^\bullet$  ~~acyclic~~ cplx of inj and  $f: C^\bullet \rightarrow I^\bullet$  any map. Then  $f \sim 0$ .

Thm: Let  $R$  be a class adapted to some functor  $F$ . Let  $S_R$  be the class of q-iso in  $K^+(R)$ . Then

$$K^+(R)[S_R^{-1}] \longrightarrow D^+(\mathcal{A})$$

is an equivalence.

Cor:  $K^+(\mathcal{A}) \cong D^+(\mathcal{A})$

ph: A quasi-iso in  $K^+(\mathcal{A})$  is an iso.

Def: A right derived functor of a left exact functor  $F$  is a pair consisting of

$$RF: D^+(\mathcal{A}) \longrightarrow D^+(\mathcal{B})$$

together with

$$\begin{array}{ccc} K^+(\mathcal{A}) & \xrightarrow{K^+(F)} & K^+(\mathcal{B}) \\ \downarrow Q_A & \searrow \epsilon_F & \downarrow Q_B \\ D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{B}) \end{array}$$

