

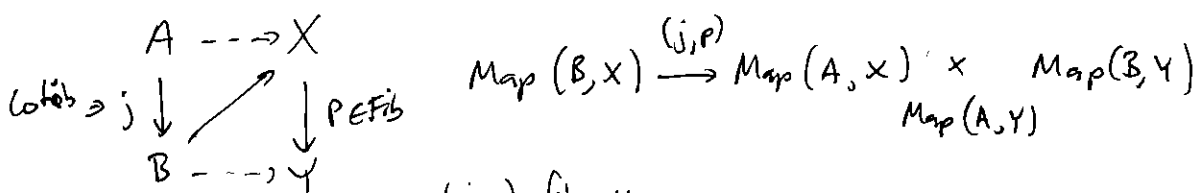
3. Hebung ± E. Algorist 23/2 - 2018

Def: A simplicial model cat is a model cat \mathcal{M} w/ $\mathcal{M} \times \mathcal{S} \xrightarrow{- \circ -} \mathcal{M}$ (kensored) ($\mathcal{S} = \text{Set}$)
 $\mathcal{M}^{op} \times \mathcal{M} \xrightarrow{\text{Map}(_, _)} \mathcal{S}$
 $\mathcal{M} \times \mathcal{S}^{op} \xrightarrow{(_)_} \mathcal{M}$ (co-kensored)

s.t. $\ast \text{Map}_{\mathcal{M}}(X, Y)_0 = \text{Hom}_{\mathcal{M}}(X, Y)$

$\ast \exists$ nat iso: $\text{Map}_{\mathcal{M}}(N, M^K) \cong \text{Map}_{\mathcal{S}}(K, \text{Map}(N, M)) \cong \text{Map}_{\mathcal{M}}(N \otimes K, M)$

\ast SM7 "corner axiom"



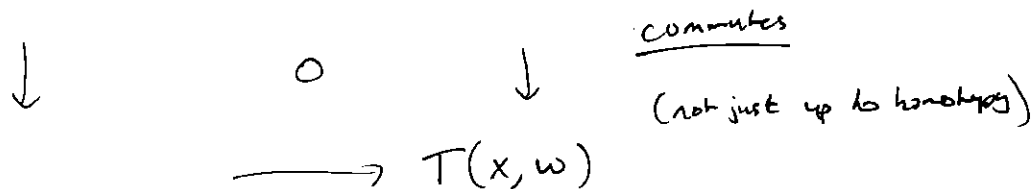
- (j, p) fibration
- and (j, p) trivial fib if j or p trivial

Ex: $\mathcal{S} \quad M \otimes K = M \times K$
 $\text{Map}(M, N) = M^N = \left(\text{Hom}(M \times \Delta[n], N) \right)_{n \geq 0}$
 \uparrow
 n -simplices

Def: A simplicial category (or ω -cat) is a category (strictly) enriched over \mathcal{S} .

i.e., \exists ass: $T(x, y) \times T(y, z) \rightarrow T(x, z)$

s.t. $T(x, y) \times T(y, z) \times T(z, w) \rightarrow$



A simplicial functor $f: T \rightarrow S$ is an \mathcal{S} -enriched functor (strict)

$$\begin{array}{ccc}
 T(x, y) \longrightarrow S(fx, fy) & \text{s.t.} & T(x, y) \times T(y, z) \longrightarrow S(fx, fy) \times S(fy, fz) \\
 \downarrow & & \downarrow \\
 T(x, z) \longrightarrow S(fx, fz) & & \downarrow \\
 & & \text{commutes}
 \end{array}$$

A natural transformation $\alpha: f \rightarrow g$ is a collection

$$(\alpha_x: f(x) \rightarrow g(x))_{x \in T} \text{ in } \mathcal{T}_0(fx, gx) := S(fx, gx).$$

s.th.

$$\begin{array}{ccccc} S(x, y) & \longrightarrow & T(fx, fy) & = & T(fx, fy) \times \Delta[0] \\ g \downarrow & \circ & \downarrow \alpha_{y*} & \searrow & T(fx, fy) \times T(fy, gy) \\ T(gx, gy) & \xrightarrow{\alpha_x^*} & T(fx, gy) & \longleftarrow & \circ \end{array}$$

Let \mathcal{M} cofibrantly generated simplicial model category, T ∞ -cat.

Then \exists model structure on $\mathcal{M}^T =$ category where
 (projective model str) (treating \mathcal{M} as a simplicial category)
 obj = simplicial functors $T \rightarrow \mathcal{M}$
 mor = nat trans

* fibration & w.c. are pointwise In fact, \mathcal{M}^T is a simplicial model cat.

Def: $SPr(T) := \mathcal{S}^{T^{op}}$ is the model category of simplicial presheaves on T .

Prop: Simplicial model cat w/

$$(F \otimes K)(x) := F(x) \times K$$

$$F^K(x) = F(x)^K$$

$$\text{Map}(F, G) = \left(\text{Hom}_{SPr(T)}(F \times \Delta[n], G) \right)_{n \geq 0}$$

Simplicial Yoneda: T ∞ -category.

Def: $x \in T$. Define simplicial functor

$$\begin{aligned} \Delta_x: T^{op} &\rightarrow \mathcal{S} \\ y &\longmapsto T(y, x) \end{aligned}$$

Lemma: $\forall F \in SPr(T), \forall x \in T: F(x) \underset{iso}{\simeq} \text{Map}_{SPr(T)}(\Delta_x, F)$

Yoneda embedding: $h: T \longrightarrow \mathcal{S}pr(T)$
 $x \longmapsto h_x$

$$T(x', x) \longrightarrow \text{Map}_{\mathcal{S}pr}(h_{x'}, h_x)$$

is Yoneda

$$\begin{matrix} h_x(x') \\ \parallel \\ T(x', x) \end{matrix}$$

So h is fully faithful (induces iso on Hom-spaces)

(works for any enriched category)

Homotopy category of an ∞ -category

Def: $H(T)$ of ∞ -cat T is:

$$\begin{aligned} \text{obj } H(T) &= \text{obj } T \\ \text{Hom}_{H(T)}(x, y) &= \pi_0 T(x, y) \end{aligned}$$

Remk: Functorial in T so we get

$$h: H(T) \longrightarrow H(\mathcal{S}pr(T))$$

Remk: \subset simplicial model cat, $f, g: X \rightarrow Y$ morphisms in \mathcal{C} . If f, g simplicial homotopic, i.e. if $f = g$ in $\pi_0(\text{Map}_{\mathcal{C}}(X, Y))$ then also $f = g$ in $\text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y)$
 ($\pi_0(\dots) = \text{Hom}_{\text{Ho}(\mathcal{C})}(X, Y)$ if X cofibrant, Y fibrant)

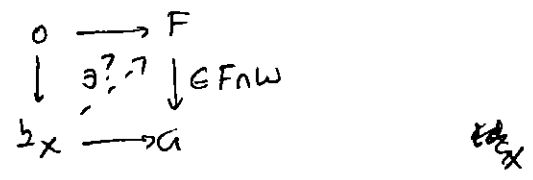
so have well-defined: $h: H(T) \longrightarrow H(\mathcal{S}pr(T)) \longrightarrow \text{Ho}(\mathcal{S}pr(T))$

Prop: Let $F \in \mathcal{SPr}(T)$, $x \in T$. Then $F(x) \simeq \mathbb{R}Map(\underline{h}_x, F)$ in $\mathcal{Ho}(\mathcal{S})$

where $\mathbb{R}Map(\underline{h}_x, F) = \mathbb{R}Map(\underline{h}_x, -)(F)$.

pt: ① \underline{h}_x is cofibrant.

Let $F \rightarrow G$ be a trivial fibration in $\mathcal{SPr}(T)$ then



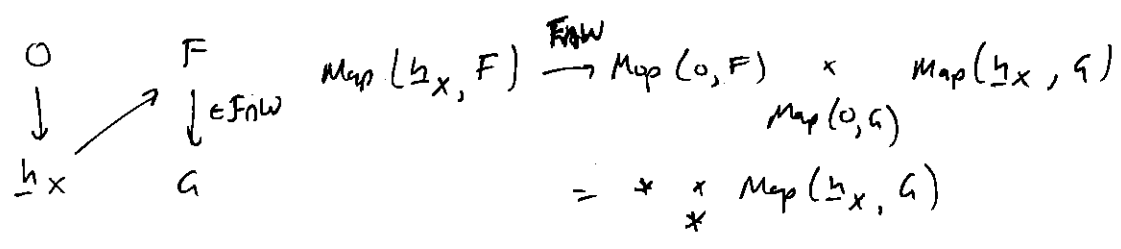
@ x : $\begin{array}{ccc} & & F_x \\ & \exists \overset{?}{\dashrightarrow} & \downarrow \in Fw \\ \underline{h}_x(x) & \longrightarrow & G_x \end{array}$ gives $\underline{h}_x \rightarrow F$ by Yoneda.
 cofibrant
 (b/c all ss. cofibrant)

② $\mathbb{Ho}Map(\underline{h}_x, -): \mathcal{SPr}(T) \rightarrow \mathcal{S} \rightarrow \mathcal{Ho}(\mathcal{S})$

sends w.e. b/w fibrant obj to iso.

By Ken Brown's lemma: (preserving initial fibrations b/w fibrant obj) \Rightarrow (preserving w.e. b/w fibrant obj)
 suffices to prove

Let $F \xrightarrow{f} G$ trivial fibration b/w fibrant objects:



Easier pt: w.e. is calculated pointwise! So $F(x) \rightarrow F(y)$ w.e.

③ Conclusion $\mathbb{R}Map(\underline{h}_x, F) = \mathbb{R}Map(\underline{h}_x, RF) \simeq (RF)(x) \simeq F(x)$
 $\mathbb{R}Map(\underline{h}_x, -)$ exists (by ②) and \uparrow by def of w.e.