DAG-seminar #4

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Def: A simplicial model cat is a model cat $\mathcal{M}$ w/ $\mathcal{M}\times\mathcal{S}\xrightarrow{-\otimes-}\mathcal{M}$ (tensored) ($\mathcal{S}$ = sSet)

$$\mathcal{M}^{op}\times\mathcal{M}\xrightarrow{Map(-,-)}\mathcal{S}$$

$$\mathcal{M}\times\mathcal{S}^{op}\xrightarrow{(-)^{-}}\mathcal{M} \quad \text{(cotensored)}$$

s.th.
* $Map_{\mathcal{M}}(X,Y)_0 = Hom_{\mathcal{M}}(X,Y)$
* $\exists$ nat iso: $Map_{\mathcal{M}}(N, M^K) \cong Map_{\mathcal{S}}(K, Map(N,M)) \cong Map_{\mathcal{M}}(N\otimes K, M)$
* SM7 "corner axiom"

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[d, "\text{Cofib} \ni j"'] & X \arrow[d, "p \in \text{Fib}"] \\ B \arrow[r, dashed] \arrow[ur] & Y \end{tikzcd} \qquad Map(B,X) \xrightarrow{(j,p)} Map(A,X) \underset{Map(A,Y)}{\times} Map(B,Y)$$

- (j,p) fibration
- and (j,p) trivial fibr if j or p trivial

Ex: $\mathcal{S}$ $\quad M\otimes K = M\times K$

$$Map(M,N) = M^N = \big(\underset{\substack{\uparrow \\ \text{n-simplicies}}}{Hom(M\times\Delta[n], N)}\big)_{n\geq 0}$$

Def: A simplicial category (or ∞-cat) is a category (strictly) enriched over $\mathcal{S}$.

i.e., $\exists$ ass: $T(x,y)\times T(y,z)\longrightarrow T(x,z)$

s.th.

$$\begin{tikzcd} T(x,y)\times T(y,z)\times T(z,w) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & {} \arrow[d] \\ {} \arrow[r] & T(x,w) \end{tikzcd}$$

commutes (not just up to homotopy)

A simplicial functor $f: T \to S$ is an $\mathcal{S}$-enriched (strict) functor

$$T(x,y) \longrightarrow S(fx,fy) \quad \text{s.th.} \quad \begin{tikzcd} T(x,y)\times T(y,z) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & S(fx,fy)\times S(fy,fz) \arrow[d] \\ T(x,z) \arrow[r] & S(fx,fz) \end{tikzcd} \quad \text{commutes.}$$

A <u>natural transformation</u> $\alpha: f \to g$ is a collection

$$(\alpha_x : f(x) \to g(x))_{x \in T} \quad \text{in } T_0(fx, gx) := S(fx, gx)_0$$

s.th.

$$\begin{tikzcd} S(x,y) \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\circ"] & T(fx,fy) = T(fx,fy) \times \Delta[0] \arrow[d, "\alpha_{y*}"] \arrow[r] & T(fx,fy) \times T(fy,gy) \arrow[dl, "\circ"] \\ T(gx,gy) \arrow[r, "\alpha_x^*"'] & T(fx,gy) & \end{tikzcd}$$

Let $\mathcal{M}$ <u>cofibrantly generated simplicial model category</u>, T ∞-cat.

Then $\exists$ model structure on $\mathcal{M}^T$ = category where obj = simplicial functors $T \to \mathcal{M}$, mor = nat trans
(<u>projective model str</u>) (treating $\mathcal{M}$ as a simplicial category)

* fibration & w.e. are pointwise. In fact, $\mathcal{M}^T$ is a <u>simplicial model cat</u>.

<u>Def</u>: $SPr(T) := S^{T^{op}}$ is the model category of simplicial presheaves on T.

<u>Rmk</u>: <u>Simplicial</u> model cat w/

$$(F \otimes K)(x) := F(x) \times K$$

$$F^K(x) = F(x)^K$$

$$\mathrm{Map}(F,G) = \left(\mathrm{Hom}_{SPr(T)}(F \times \Delta[n], G)\right)_{n \geq 0}$$

<u>Simplicial Yoneda</u>: T ∞-category.

<u>Def</u>. $x \in T$. Define simplicial functor

$$\underline{h}_x : T^{op} \to S$$
$$y \mapsto T(y,x)$$

<u>Lemma</u>: $\forall F \in SPr(T), \forall x \in T$: $F(x) \underset{iso}{\simeq} \mathrm{Map}_{SPr(T)}(\underline{h}_x, F)$

Yoneda embedding: $\underline{h} : T \longrightarrow SPr(T)$

$$x \longmapsto \underline{h}_x$$

$$T(x', x) \longrightarrow Map_{SPr}(\underline{h}_{x'}, \underline{h}_x)$$
$$\wr\!\wr \text{ Yoneda}$$
$$\underline{h}_x(x')$$
$$\|$$
$$T(x', x)$$

So $\underline{h}$ is fully faithful (induces iso on Hom-spaces)

(works for any enriched category)

Homotopy category of an ∞-category

Def: $H(T)$ of ∞-cat T is:

$$\underline{\text{obj}}\, H(T) = \text{obj}\, T$$
$$Hom_{H(T)}(x, y) = \pi_0 T(x, y)$$

Rmk: Functorial in T so we get

$$\underline{h} : H(T) \longrightarrow H(SPr(T))$$

Rmk: $\mathcal{C}$ simplicial model cat, $f, g : X \to Y$ morphisms in $\mathcal{C}$. If f, g simplicial homotopic, i.e. if $f = g$ in $\pi_0(Map_{\mathcal{C}}(X, Y))$ then also $f = g$ in $Hom_{Ho(\mathcal{C})}(X, Y)$

($\pi_0(-) = Hom_{Ho\mathcal{C}}(X, Y)$ if X cofibrant, Y fibrant)

so have well-defined: $h : H(T) \longrightarrow H(SPr(T)) \longrightarrow Ho(SPr(T))$

Prop: Let $F \in \mathcal{S}Pr(T)$, $x \in T$. Then $F(x) \simeq \mathbb{R}Map(\underline{h}_x, F)$ in $Ho(\mathcal{S})$ where $\mathbb{R}Map(\underline{h}_x, F) = \mathbb{R}Map(\underline{h}_x, -)(F)$.

pf: ① $\underline{h}_x$ is cofibrant.

Let $F \to G$ be a trivial fibration in $\mathcal{S}Pr(T)$ then

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F \arrow[d, "\in F\cap W"] \\ \underline{h}_x \arrow[r] \arrow[ur, dashed, "\exists ?"] & G \end{tikzcd}$$

@ x:

$$\begin{tikzcd} & F_x \arrow[d, "\in F\cap W"] \\ \underline{h}_x(x) \arrow[r] \arrow[ur, dashed, "\exists"] & G_x \end{tikzcd}$$

gives $\underline{h}_x \longrightarrow F$ by Yoneda.

cofibr (b/c all s.s. cofibrant)

② $Ho\,Map(\underline{h}_x, -) : \mathcal{S}Pr(T) \longrightarrow \mathcal{S} \to Ho(\mathcal{S})$

sends w.e. b/w fibrant obj to iso.

[By Ken Brown's lemma: suffices to prove (preserving trivial fibrations b/w fibrant objs) $\Rightarrow$ (preserving w.e. b/w fibrant obj)

Let $F \xrightarrow{f} G$ trivial fibration b/w fibrant objects:

$$\begin{tikzcd} 0 \arrow[d] & F \arrow[d, "\in F\cap W"] \\ \underline{h}_x \arrow[ur] & G \end{tikzcd}$$

$$Map(\underline{h}_x, F) \xrightarrow{F\cap W} Map(0,F) \times_{Map(0,G)} Map(\underline{h}_x, G) = * \times_{*} Map(\underline{h}_x, G)$$
]

Easier pf: w.e. is calculated pointwise! So $F(x) \longrightarrow F(y)$ w.e.

③ Conclusion $\mathbb{R}Map(\underline{h}_x, F) = \mathbb{R}Map(\underline{h}_x, RF) \simeq (RF)(x) \underset{\substack{\uparrow \\ \text{by def of w.e.}}}{\simeq} F(x)$

$\mathbb{R}Map(\underline{h}_x, -)$ exists (by ②) and