DAG-seminar #3

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① Model structures on $\mathcal{S}$

② Derived functors

③ Simplicial (model) categories

① Model structures on $\mathcal{S}$

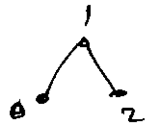
Standard model structure (Quillen model structure)

$f : X \longrightarrow Y$ in $\mathcal{S}$ is a

* weak equivalence if $|f| : |X| \longrightarrow |Y|$ is a weak eq.

* fibration if it's a Kan fibration, i.e.

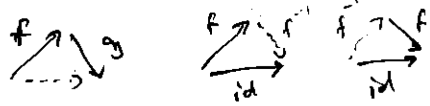
$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

Ex: $\Lambda^1[2] =$ 

$\Lambda^k[n]$

* cofibration if it's an injection

Rmk: A fibrant simplicial object is a Kan complex.

A Kan complex is a model for a ∞-groupoid. (associativity, inverses: )

Rmk: For Kan complexes, we have a purely comb. description of $\pi_n(K, x_0)$ where elements are repr by

$$\alpha : \Delta[n] / \partial\Delta[n] \longrightarrow K$$

modulo equivalence relation (is an eq rel. b/c Kan cplx)

$$\begin{tikzcd} \Delta[n] \times \Delta\{0\} \arrow[d] \arrow[dr, "\alpha"] & \\ \Delta[n] \times \Delta[1] \arrow[r] & K \\ \Delta[n] \times \Delta\{1\} \arrow[u] \arrow[ur, "\alpha'"'] & \end{tikzcd}$$

$\Delta[1]$ is a cylinder obj.

② Derived functors

Def: Let $F: M \to \mathcal{D}$, M model category. A left derived functor of F is:

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\ Ho(M) \arrow[ur, dashed, "LF"'] \arrow[u, phantom, shift left=4, "t\Uparrow"] & \end{tikzcd}$$

s.th $\forall$

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\ Ho(M) \arrow[ur, "G"] \arrow[ur, bend right=50, "LF"'] \arrow[u, phantom, shift left=4, "s\Uparrow"] & \end{tikzcd} \qquad \begin{tikzcd} G\gamma \arrow[rr, Rightarrow, "u\gamma"] \arrow[dr, Rightarrow, "s"'] & \circ & LF\gamma \arrow[dl, Rightarrow, "t"] \\ & F & \end{tikzcd}$$

$\exists ! u$

It's the Right Kan extension of F.

Ex: Suppose $F: M \to D$ homotopical (i.e. weak eq $\mapsto$ iso), then since γ localization, we get

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\ Ho(M) \arrow[ur, "LF"'] \arrow[u, phantom, shift left=4, "id\Uparrow"] & \end{tikzcd}$$

left der. functor

pf

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\ Ho(M) \arrow[ur, "G"'] \arrow[u, phantom, shift left=4, "s\Uparrow"] & \end{tikzcd}$$

Let $(G \overset{u}{\Rightarrow} LF)$: ~~G~~ where $u_{\gamma X}: G\gamma(X) \xrightarrow{s_X} FX = LF\gamma X$.

Rmk: Right derived functors: same but $\exists ! u: RF \Rightarrow G$.

Prop: Let $F: M \to D$ be such that it maps w.eq. b/w cofibrant obj's to isos.

Then F has a left derived functor and $t_X: LFX \to FX$ identity if X cofibrant.

pf:

$$\begin{tikzcd} M \arrow[r, "Q"'] \arrow[r, bend left=60, "id"] \arrow[r, phantom, bend left=30, "\Uparrow p"] \arrow[d, "\gamma"'] & M \arrow[r, "F"] & D \\ Ho(M) \arrow[urr, "LF"'] \arrow[u, phantom, shift left=6, "\Uparrow id"] & & \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "Q"'] \arrow[r, bend left=60, "id"] \arrow[r, phantom, bend left=30, "\Uparrow"] & M \end{tikzcd}$$ cofibrant repl.

Let $t = Fp \circ id \Rightarrow$

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\ Ho(M) \arrow[ur, "LF"'] \arrow[u, phantom, shift left=4, "\Uparrow t"] & \end{tikzcd}$$

Claim: t makes LF left deriv. functor.

(use that p invertible in $Ho(M)$).

<u>Def</u>: Let

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d, "\gamma"'] & N \arrow[d, "\delta"] \arrow[dl, phantom, "\Nearrow"] \\ Ho(M) \arrow[r, "\mathbb{L}F"] & Ho(N) \end{tikzcd}$$

be a functor b/w model categories. A <u>total left der. functor</u> is a $\cancel{\mathbb{L}}\mathbb{L}F$ ~~fitting~~ in the diagram to the left.

<u>Thm</u> (Quillen's total derived functor thm)

(1) Let $F: M \rightleftarrows N : G$ be an adjunction s.th either

- F preserves cofibr, G preserves fibrations; or
- F preserves cofibr + trivial cofibr; or
- G preserves fibr + trivial fibr; or
- F preserves w.eq b/w M_c = cofibr obj of M
  <u>and</u> G — " — N_f = fibr obj of N

Then $\exists\ \mathbb{L}F : M \rightleftarrows N : \mathbb{R}G$ and form an <u>adjunction</u>.

(2) Furthermore if $(FA \to X) \in W \Leftrightarrow (A \to GX) \in W \quad \forall A \in M_c,\ X \in N_f$

Then $\mathbb{L}F, \mathbb{R}G$ <u>adjoint equiv. of cats</u>.

We say F, G Quillen adjunction / equiv.

<u>Ex</u>: $|\cdot| : S \rightleftarrows Top : \overset{S_{\cdot}}{Sing}$ is a Quillen eq.

<u>Claim</u>: For a space X, the counit $|SX| \to X$ is a w.eq:

$$\alpha \in \pi_n(X,x) \leftrightarrow (\Delta^n \to X) = (|\Delta[n]| \to X) \overset{adj}{\leftrightarrow} (\Delta[n] \to SX)$$

$$\updownarrow$$

$$\underset{\text{comb. desc.}}{\pi_n(|SX|, x) \simeq \pi_n(SX, x)} \qquad\qquad \beta \in \pi_n(SX, x)$$

* If $f : X \to Y$ w.e in Top $\Rightarrow$ Sf w.e.

$$\begin{tikzcd} X \arrow[r, "f", "\sim"'] & Y \\ {|SX|} \arrow[r] \arrow[u, "s", leftrightarrow] & {|SY|} \arrow[u, "s"', leftrightarrow] \end{tikzcd} \Rightarrow |Sf| \text{ w.e.} \overset{\text{def.}}{\Rightarrow} Sf \text{ w.e.}$$

Proves that we a Quillen adjunction.

* For $A \in \mathcal{S}$, $X \in \mathcal{T}op$. $\left(A \underset{\text{w.e.}}{\xrightarrow{\sim}} SX\right) \overset{\text{by def}}{\Longleftarrow} \left(|A| \xrightarrow{\sim} |SX|\right) \overset{\text{Claim}}{\Longleftrightarrow} \left(|A| \xrightarrow{\sim} X\right)$

so Quillen equiv.

(Quillen Annals recent paper :

Cochain complexes

R ring, $M := Ch^*_{\geq 0}(R) = \{ C^0 \xrightarrow{d^0} C^1 \to \dots \}$

model structure: (injective model structure)

* weak eq : q-iso's
* fibrations: surjections w/ injective kernels
* cofibrations: ~~injective~~ maps that are injective in degrees ≥ 1 !

Rmk: Fibrant obj's are complexes of injectives. $I^0 \to I^1 \to \dots$

Take M an R-module, let $M[0] \to I$ be a fibrant replacement.

$$\text{This is} \quad \begin{tikzcd} M \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \dots \\ I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \dots \end{tikzcd}$$

trivial cofibr $\to$

q.iso $\Rightarrow M = \ker(I^0 \to I^1)$

so $M \to I^0$ inj.

$$\mathrm{Hom}_R(M,-) : \underset{\substack{\| \\ Ch^*_{\geq 0} R}}{\mathcal{M}} \longrightarrow \underset{\substack{\| \\ Ch^*_{\geq 0} \mathbb{Z}}}{\cancel{Ch}_{\geq 0} \mathcal{N}}$$

By Prop 1, it has a total right derived functor

$$\mathbb{R}\mathrm{Hom}(M,-) : \mathrm{Ho}(\mathcal{M}) \longrightarrow \mathrm{Ho}(\mathcal{N})$$

$$\mathcal{H}^i \mathbb{R}\mathrm{Hom}(M, N[0]) = \mathcal{H}^i \mathrm{Hom}(M, I^\bullet) =: \mathrm{Ext}^i(M,N)$$

Think of simplicial sets as categories:

obj: $\Delta[n] \xrightarrow{\sigma} K$

mor: expl $\left(\begin{tikzcd} \Delta[n+1] \arrow[r, "s_i"] \arrow[dr] & \Delta[n] \arrow[d] \\ & K \end{tikzcd}\right)$ ~~and~~ any $\begin{tikzcd} \Delta[m] \arrow[rr] \arrow[dr, "\sigma"'] & & \Delta(n) \arrow[dl, "\sigma'"] \\ & K & \end{tikzcd}$