DAG-seminar #2

J. Hekking 9/2-2018

Model structures

① Def of model categories
② $\mathcal{T}op$ w/ Quillen model structure
③ Homotopy cat.
④ sSet
⑤ Derived functors
⑥ Cochain complexes

① Def of M.C

Def. A model category $\mathcal{M}$ is a category w/ 3 classes of morphisms:

W: weak eq
$\mathcal{C}$: cofibrations
$\mathcal{F}$: fibrations

s.t.

(i) $\mathcal{M}$ finitely complete + cocomplete
(ii) W contains all iso and satisfies 3 for 2
(iii) $(\mathcal{C}, \underbrace{W \cap \mathcal{F}}_{\text{trivial fibr}})$ and $(\underbrace{\mathcal{C} \cap W}_{\text{trivial cofibr}}, \mathcal{F})$ weak fact. systems

Def: $(\mathcal{L}, \mathcal{R})$ WFS if $\forall\, X \xrightarrow{f} Y$ $\exists$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\in \mathcal{L}"'] & & Y \\ & Z \arrow[ur, "\in \mathcal{R}"'] & \end{tikzcd}$$

and $\mathcal{L} = \{\text{maps w/ LLP w.r.t. } \mathcal{R}\}$
$\mathcal{R} = \{\text{maps w/ RLP w.r.t. } \mathcal{L}\}$

Nice if factorization is functorial (some require it, Hovey?)

$\in \mathcal{R}$

$\Rightarrow \in \mathcal{L}$ etc

Lem: If $(\mathcal{L}, \mathcal{R})$ WFS, then $\mathcal{L}$ & $\mathcal{R}$ contain all isos, closed under comp & retracts and $\mathcal{L}$ closed under pullback and $\mathcal{R}$ closed under pushouts.

Lem (Joyal) $\mathcal{M}$ model cat. If f retract of g then

$$g \in \mathcal{W} \text{ (or } \mathcal{C} \text{ or } \mathcal{F}) \text{ then } f \in \mathcal{W} \text{ (or } \mathcal{C} \text{ or } \mathcal{F})$$

pf: Previous lemma gives $\mathcal{C}/\mathcal{F}$ case. Suppose $g \in \mathcal{W}$.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] \arrow[rr, bend left, "id"] & X \arrow[r] \arrow[d, "g"', "\sim"] \arrow[dr, "\mathcal{C}\cap\mathcal{W}"] & A \arrow[d, "f"] \\
B \arrow[r] & Y \arrow[r] & B \\[-2em]
& &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[d, "\mathcal{C}\cap\mathcal{W}"] \arrow[r] & A \\
Z \arrow[d, "\mathcal{F}"] \arrow[r, dashed] \arrow[dr] & \\
Y \arrow[r] & B
\end{tikzcd}$$

Suppose f fibration. Then get $Z \dashrightarrow A$ so f retract of $Z \to Y$ which is $\mathcal{F} \cap \mathcal{W}$

$\Rightarrow f \in \mathcal{F} \cap \mathcal{W}$.

In general:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"', "\mathcal{W}\cap\mathcal{C}"] \arrow[dd, bend right=60, "f"'] & X \arrow[r] \arrow[d, "\mathcal{W}\cap\mathcal{C}"] & A \arrow[d] \\
E \arrow[r, "i"] \arrow[d, "p"', "\mathcal{F}"] & E \amalg_A X \arrow[r, "r"] \arrow[d, "\mathcal{W}"] & E \arrow[d, "\mathcal{F}"] \\
B \arrow[r] & Y \arrow[r] & B
\end{tikzcd}$$

previous case $\Rightarrow p$ w.e.

Def. $A \in \mathcal{M}$. A is <u>cofibrant</u> if $(0 \to A) \in \mathcal{C}$

A is <u>fibrant</u> if $(A \to 1) \in \mathcal{F}$

(co)Fibrant replacements

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "C"'] & & X \\ & QX \arrow[ur, "F\cap W"'] & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr] \arrow[dr, "C\cap W"'] & & 1 \\ & RY \arrow[ur, "F"'] & \end{tikzcd}$$

cofibrant replacement   fibrant repl.

② $\mathcal{T}op$

Ex: $\mathcal{T}op$ model category w/

- w.e. are weak homotopy equiv.
- cofibrations are the retracts of relative cell complexes $X \longrightarrow Y$.
  (Y is obtained from X by attaching cells)
- fibrations are the Serre fibrations:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ I^n \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

Rmk:

$$\begin{tikzcd} I^n \times \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ I^n \times I \arrow[r] \arrow[ur, dashed, "\exists"] & * \end{tikzcd}$$

for any X. so all X fibrant.

Cofibrant:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r] & B \arrow[r] & X \end{tikzcd}$$

$\Rightarrow A = 0$ so X retract of cell complex.

③ Homotopy cat

<u>Def</u>: A <u>Cylinder object</u> of A is: $A \wedge I$ given by:

$$\begin{tikzcd} A \amalg A \arrow[rr, "\nabla"] \arrow[dr, "i", "C"'] & & A \\ & A \wedge I \arrow[ur, "\sigma", "F \cap W"'] & \end{tikzcd}$$

A <u>left homotopy</u> $(f \overset{l}{\sim} g) : A \to B$ is

$$\begin{tikzcd} A \amalg A \arrow[rr, "{(f,g)}"] \arrow[dr, "i"'] & & B \\ & A \wedge I \arrow[ur, "h"'] & \end{tikzcd}$$

A <u>path object</u> for B is B^I:

$$\begin{tikzcd} B \arrow[rr, "\Delta"] \arrow[dr, "s", "C \cap W"'] & & B \times B \\ & B^I \arrow[ur, "p", "F"'] & \end{tikzcd}$$

A <u>right homotopy</u> $(f \overset{r}{\sim} g) : A \to B$

$$\begin{tikzcd} A \arrow[rr, "{(f,g)}"] \arrow[dr, "h"'] & & B \times B \\ & B^I \arrow[ur, "p"'] & \end{tikzcd}$$

<u>Ex</u> in Top

$$\begin{tikzcd} A \amalg A \arrow[d] \arrow[dr] & \\ A \times I \arrow[r, "h"] & B \end{tikzcd}$$

cofibrant if A <u>cell cplx</u>

<u>Prop</u>: A cofibrant, B fibrant. Then

$(f \overset{l}{\sim} g) \Leftrightarrow (f \overset{r}{\sim} g)$

and we write $f \sim g$.

Def: The homotopy category $Ho(\mathcal{M})$ of $\mathcal{M}$ is the category w/ same objects as $\mathcal{M}$ and:

$$Ho(\mathcal{M})(X,Y) = \mathcal{M}(RQX, RQY)/\sim$$

Thm: There is a functor $\gamma : \mathcal{M} \to Ho(\mathcal{M})$ which is the identity on objects making $Ho(\mathcal{M})$ localization of $\mathcal{M}$ at $\mathcal{W}$. Universal:

$$\begin{tikzcd} \mathcal{M} \arrow[r, "W \mapsto \text{iso}"] \arrow[d, "\gamma"'] & D \\ Ho(\mathcal{M}) \arrow[ur, dashed, "\exists !"'] & \end{tikzcd}$$

and $f : X \to Y$ w.e. in $\mathcal{M}$ $\Leftrightarrow$ γf iso.

pf:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, "c"'] & QY \arrow[d, "F\cap W"] \\ QX \arrow[r] \arrow[ur, dashed, "Qf"] \arrow[d, "F\cap W"'] & Y \\ X \arrow[ur, "f"'] & \end{tikzcd} \qquad \begin{tikzcd} & QY \arrow[dr, "C\cap W"] & \\ QX \arrow[ur, "Qf"] \arrow[rr] \arrow[d, "C\cap W"'] & & RQY \arrow[d, "F"] \\ RQX \arrow[urr, dashed, "RQf"] \arrow[rr, "F"'] & & * \end{tikzcd}$$

$$\gamma f := [RQf]_\sim$$

Different choices of Qf will give left homotopy ($F_1 \overset{\ell}{\sim} F_2$)
F_1, F_2

Similarly diff choices of lifts RQf $\rightsquigarrow$ right homotopy.

Rmk: RQX is both fibrant and cofibrant.

<u>Interlude</u>: Let $f : X \longrightarrow Y$ map of top spaces

(1) $\forall n$: $[S^n, f] : [S^n, X] \longrightarrow [S^n, Y]$ bij

(2) $\forall K$ fin CW: $[K, f]$ bij

(3) $\forall K$ CW: $[K, f]$ bij

(4) $\forall n \, \forall x \in X \, [S^n, f]_*$ bij

(5) $\forall K$ fin CW $\forall x \in X$ $[K, f]_*$ bij

(6) $\forall K$ CW $\forall x \in X$ $[K, f]_*$ bij

Then $(1) \Leftrightarrow (2) \Leftarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6)$

<u>Counterex</u>: $\Sigma_\infty = \{ \sigma : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ identity on all but fin many $\}$ discrete group

$$\Sigma_\infty \hookrightarrow \Sigma_\infty$$

$$\sigma \longmapsto \begin{pmatrix} 0 \longmapsto 0 \\ n \longmapsto \sigma(n-1)+1 \quad \forall n > 0 \end{pmatrix}$$

Then $B\Sigma_\infty \xrightarrow{f} B\Sigma_\infty$ counterexample.

This reflects that you have to work when proving that $\mathcal{T}op$ is a model structure.

④ Simplicial sets

<u>Def</u>: Δ simplex category:

objs Δ: $[n] = \{0,1,\dots,n\}$ $\quad (n\geq 0)$

special maps: $d^i: [n-1] \longrightarrow [n]$, $\quad (0,\dots,n-1) \longmapsto (0,\dots,\hat{i},\dots,n)$

$0\leq i,j\leq n \quad s^j: [n+1] \longrightarrow [n]$, $\quad (0,\dots,n+1) \longmapsto (0,\dots,j,j,\dots,n)$

generate all maps

<u>Def</u>. A simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{op} \xrightarrow{X} \mathcal{C}$.

So such an object is
$$\begin{tikzcd} X_0 \arrow[r, "s_0"] & X_1 \arrow[l, bend right, "d_0"'] \arrow[l, bend left, "d_1"] \arrow[r, shift left=2] \arrow[r, shift right=2] & X_2 \arrow[l, shift left=1] \arrow[l, shift right=3] & \cdots \end{tikzcd}$$

<u>Def</u>: $\Delta[-]: \Delta \longrightarrow \overset{sSet}{\overset{\shortparallel}{\mathcal{S}}} := Set^{\Delta^{op}}$

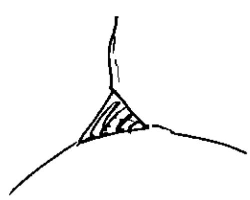
$\qquad [n] \longmapsto \Delta(-,[n])$

<u>Rmk</u>: $X \in \mathcal{S}$, then $X_n \overset{bij}{\simeq} \mathcal{S}(\Delta[n], X)$

$$X \cong \operatorname*{colim}_{\Delta[n]\to X} \Delta[n]$$

<u>Geometric realization</u> of a sSet X:

$$|X| := \operatorname*{colim}_{\Delta[n]\to X} \Delta^n \qquad \Delta^n \subseteq \mathbb{R}^{n+1} \text{ std } n\text{-simplex}$$



Singular complex

Y space

$$SY : \Delta^{op} \longrightarrow Set$$
$$[n] \longmapsto \mathcal{T}op(\Delta^n, Y)$$

Prop: $\exists$ adjunction $\mathcal{T}op(|X|, Y) \simeq \mathcal{S}(X, SY)$

pf ~~(sketch)~~ $\mathcal{T}op(|X|, Y) = \mathcal{T}op(\operatorname{colim}_{\Delta[n] \to X} \Delta^n, Y) = \lim_{\Delta[n] \to X} \mathcal{T}op(\Delta^n, Y)$

$$= \lim_{\Delta[n] \to X} \mathcal{S}(\Delta[n], SY) = \mathcal{S}(\operatorname{colim}_{\Delta[n] \to X} \Delta[n], SY) = \mathcal{S}(X, SY)$$

(very nice paper on s.sets by Vaerodst? in J. of K-theory ~ 08)