DAG - seminar #15

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History of homological algebra

1) Cartan - Eilenberg, Godement
2) Tohoku (Grothendieck): unified treatment, abelian categories
3) Verdier's thesis: construction of derived categories, axioms of triangulated categories

Problems w/ triangulated categories

- limits/colimits badly behaved
- no realization functor: given $(\mathcal{T}, t)$ triangulated category w/ t-structure $\exists^? \; D^b(\mathcal{T}^\heartsuit) \to \mathcal{T}$
- an algebra obj in $\mathcal{T}$ $\overset{?}{\rightsquigarrow}$ triangulated category of A-modules ?
- filtered objects

Various suggested fixes:

- dg-categories (Kapranov, Beilinson, Bondal, Orlov, Keller, ...)
- derivators (Grothendieck, Alex Heller)
- A_∞-categories (chain complexes up to coherent homotopy)

4) Stable ∞-categories !

Stable ∞-categories

$\mathcal{C}$ stable ∞-cat $\rightsquigarrow$ $h\mathcal{C}$ triangulated category

Definition of ∞-cats around since 80's
Developed by Joyal and Lurie.
stable ∞-cats due to Lurie.

(stable ∞-cat) $\xrightarrow{\text{forget}}$ (stable derivators)

$\left(\begin{array}{l}\text{pretriangulated ?}\\ \text{dg-cat}\end{array}\right) \xrightarrow{N_{dg}}$ (stable ∞-cat)

$\cap$

(spectral categories) $\xrightarrow{\sim}$

(stable model categories) $\xrightarrow{\text{presentable}}$ (stable ∞-cat)

stable ∞-cats have $\otimes$-product

Limits & colimits in ∞-categories

Lurie: can do category theory in ∞-cats.

- Define join of $\mathcal{C}$ and $\mathcal{D}$: objs: ob $\mathcal{C}$ $\amalg$ ob $\mathcal{D}$
  $\mathcal{C} \star \mathcal{D}$ mor: Mor $\mathcal{C}$ $\amalg$ Mor $\mathcal{D}$ + unique map from each $c \in \mathcal{C}$ to each $d \in \mathcal{C}$
- Define slice in terms of join using adjunction: $\mathrm{Fun}^{(p)}(\mathcal{D} \star I, \mathcal{C}) \cong \mathrm{Fun}(\mathcal{D}, \mathcal{C}_{/F})$

  (1-categorically: $F : I \to C$, slice cat C/F has objs (x,c,f) $x \in I$, $c \in C$, $f : c \to F(x)$)

- Define terminal object: $c \in \mathcal{C}$ terminal if $\mathcal{C}_{/c} \to \mathcal{C}$ is a trivial fibration

  Prop: full subcat of $\mathcal{C}$ spanned by

- $\lim F$ = terminal object of $\mathcal{C}_{/F}$

Examples in ∞-cat of spaces

1)
$$\begin{tikzcd} & pt \arrow[d] \\ pt \arrow[r] & X \end{tikzcd} \xrightarrow[\text{w/ a fibration}]{\text{need to replace one}} \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & PX = \mathrm{Hom}((I,p),(X,x)) \arrow[d] \\ pt \arrow[r] & X \end{tikzcd}$$

2)
$$\begin{tikzcd} S^n \arrow[r] \arrow[d] & pt \\ pt & \end{tikzcd} \quad \begin{matrix}\text{need to} \\ \text{replace one} \\ \text{w/ a cofibration}\end{matrix} \quad \begin{tikzcd} S^n \arrow[r, hook] \arrow[d] & D^{n+1} \arrow[d] \\ pt \arrow[r] & S^{n+1} = \Sigma S^n \end{tikzcd}$$

In general
$$\begin{tikzcd} X \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & \Sigma X \end{tikzcd}$$

3) Fibers and cofibers
$$\begin{tikzcd} & X \arrow[d] \\ pt \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \\ pt & \end{tikzcd}$$

Def: An ∞-category is pointed if it has an object that is both initial and terminal. (called 0)

Def: A triangle is a diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

It is exact if it's a pullback, coexact if it's a pushout.

<u>Def</u>: Let $f: X \to Y$, map in ∞-cat.

A <u>kernel</u> to f is an exact triangle

$$\begin{tikzcd} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

A <u>cokernel</u> to f is a coexact triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

<u>Def</u>: An ∞-category is <u>stable</u> if

- it's <u>pointed</u>
- <u>kernels</u> and <u>cokernels</u> exist
- exact $\Leftrightarrow$ coexact for triangles

<u>Rmk</u>: <u>No</u> extra structure! (as triangulated cat's have)

<u>Fact</u>: $\mathcal{C}$ $\overset{\text{stable}}{\infty\text{-cat}}$ $\Rightarrow$ $h\mathcal{C}$ triangulated. Let's briefly discuss this:

1) $\operatorname{coker}(X \to 0) = \Sigma X$:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

2)

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & W \simeq \Sigma X \arrow[d] \\ & 0 \arrow[r] & V \simeq \Sigma Y \end{tikzcd}$$

$\rightsquigarrow$ exact triangle $X \to Y \to Z \to \Sigma X$

$\rightsquigarrow$ exact triangle $Y \to Z \to \Sigma X \to \Sigma Y$

How to construct a stable ∞-category?

$$N_{dg} : (dg\text{-cat}) \longrightarrow (\infty\text{-cat})$$

0-cell of $N_{dg}(C)$: object of C

1-cell ~~of~~ $X \xrightarrow{f} Y$: $f \in Hom(X,Y)_0$ s.th $df = 0$.

2-cell ~~of~~
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$
$f \in Hom(X,Y)_0$, $g \in Hom(Y,Z)_0$, $h \in Hom(X,Z)_0$, $w \in Hom(X,Z)_1$

$df = dg = dh = 0$

$dw = g \circ f - h$

(N_{dg} doesn't see negative cohomology groups). ~~whereas~~ $N_{dg}\left(\begin{smallmatrix}\text{pretriangulated}\\ \text{dg-cat}\end{smallmatrix}\right)$ is a stable ∞-cat

$$\begin{tikzcd} \left(\begin{smallmatrix}\text{pretr}\\ \text{dg-cat}\end{smallmatrix}\right) \\ \cap \\ \left(\begin{smallmatrix}\text{pretr}\\ \text{spectral cat}\end{smallmatrix}\right) \arrow[r, shift left] & (\text{stable } \infty\text{-cat}) \arrow[l, shift left] \\ \cup & \cup \\ (\text{sSet-cat}) \arrow[r, shift left] & (\infty\text{-cat}) \arrow[l, shift left] \end{tikzcd}$$

Derived categories

$\mathcal{A}$ abelian category

$Ch^?(\mathcal{A})$ dg-cat $\quad ? \in \{\emptyset, +, -, b\}$

$ho\, N_{dg}\, Ch^?(\mathcal{A}) = K^?(\mathcal{A})$

Localization Construction

$AC :=$ full subcat of $N_{dg}\, Ch^?(\mathcal{A})$ spanned by acyclic complexes

$D(\mathcal{A}) :=$ cofiber of $AC \longrightarrow N_{dg}\, Ch^?(\mathcal{A})$ (in ∞-cat of (stable?) ∞-cat's)

Alternatively: If $\mathcal{A}$ has enough injectives: $D^+(\mathcal{A}) \simeq N_{dg} Ch^+(I)$

$D(\mathcal{A}) \simeq N_{dg} Ch(K\text{-injectives})$

etc

<u>Def</u>: A <u>t-structure</u> on a stable ∞-cat $\mathcal{C}$ is a t-structure on $h_o\mathcal{C}$.

$\Rightarrow$ ∞-subcat $\mathcal{C}_{\geq n}$, $\mathcal{C}_{\leq n}$

$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$ is equivalent to $N(h_o\mathcal{C}^{\heartsuit})$

Let $\mathcal{A}$ abelian category w/ enough projectives

Let $\mathcal{C}$ stable ∞-category w/ t-structure.

$$\mathrm{REx}(\mathcal{A}, \mathcal{C}^{\heartsuit}) \simeq \mathrm{Fun}^{R.t-Ex}(\mathcal{D}^-(\mathcal{A}), \mathcal{C})^{(*)}$$

(*) means: proj obj in $\mathcal{A}$ are mapped to $\mathcal{C}^{\heartsuit}$

<u>Ex</u>: $\mathcal{A} = \mathcal{C}^{\heartsuit}$ $\Rightarrow \mathcal{D}^-(\mathcal{C}^{\heartsuit}) \to \mathcal{C}$ realizable

<u>Ex</u>: $\mathcal{A} = (Ab)$, $\mathcal{C} = (\text{Spectra})$ $\Rightarrow \mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$.

$\overset{\shortparallel}{\mathcal{C}^{\heartsuit}}$

<u>Filtered objects</u>

<u>Def</u>: A <u>filtered obj in $\mathcal{C}$</u> is a functor $N(\mathbb{Z}) \to \mathcal{C}$.

$$\begin{array}{ccl} X(n) & \longrightarrow & X(n+1) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & Y(n+1) = \mathrm{cofib}\; X(n) \to X(n+1) \end{array}$$