

History of homological algebra

- 1) Cartan - Eilenberg, Godement
- 2) Tohoku (Grothendieck): unified treatment, abelian categories
- 3) Verdier's thesis: construction of derived categories, axioms of triangulated categories

Problems w/ triangulated categories

- limits/colimits badly behaved
- no realization functor: given  $(\mathcal{T}, t)$  triangulated category w/ t-structure  $\exists? D^b(\mathcal{T}^\heartsuit) \rightarrow \mathcal{T}$
- an algebra obj in  $\mathcal{T}$   $\overset{?}{\rightsquigarrow}$  triangulated category of  $A$ -modules?
- filtered objects

Various suggested fixes:

- dg-categories (Kapranov, Beilinson, Bondal, Orlov, Keller, ...)
- derivators (Grothendieck, Alex Heller)
- $A_{\infty}$ -categories (chain complexes up to coherent homotopy)

4) Stable  $\infty$ -categories!

Stable  $\infty$ -categories

$\mathcal{C}$  stable  $\infty$ -cat  $\rightsquigarrow$  ho  $\mathcal{C}$  triangulated category

(stable  $\infty$ -cat)  $\xrightarrow{\text{forget}}$  (stable derivators)

(pretriangulated? dg-cat)  $\xrightarrow{N_{dg}}$  (stable  $\infty$ -cat)

(spectral category)  $\nearrow$

(stable model category)  $\xrightarrow{\text{premb.}}$  (stable  $\infty$ -cat)

stable  $\infty$ -cats have  $\otimes$ -product

of  $\infty$ -cats

Definition around since 80's  
Developed by Joyal and Lurie.  
stable  $\infty$ -cats due to Lurie.

Limits & colimits in  $\infty$ -categories

Limit: can do category theory in  $\infty$ -cats.

- Define join of  $\mathcal{C}$  and  $\mathcal{D}$ :  $obj: \mathcal{C} \amalg \mathcal{D}$   
 $mor: Mor_{\mathcal{C}} \amalg Mor_{\mathcal{D}} + \text{unique map from each } c \in \mathcal{C}$   
 $\text{to each } d \in \mathcal{D}$

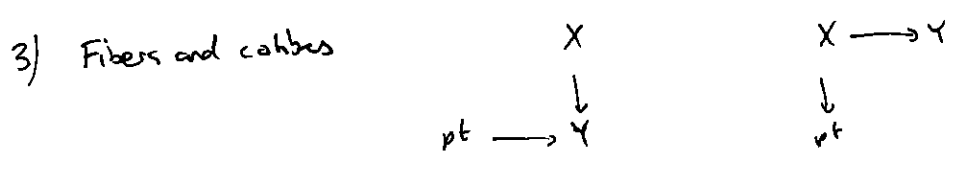
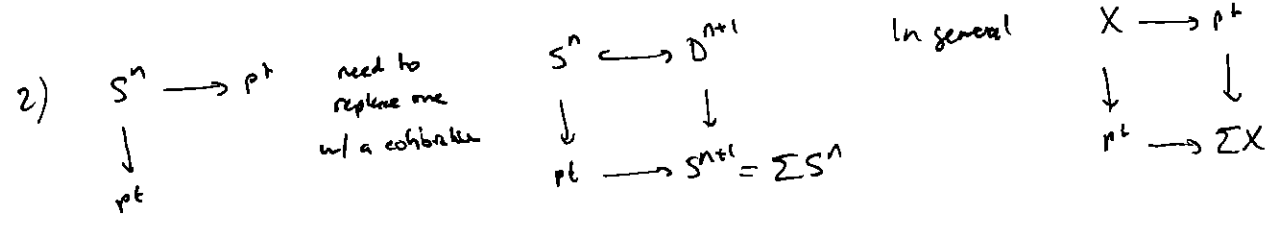
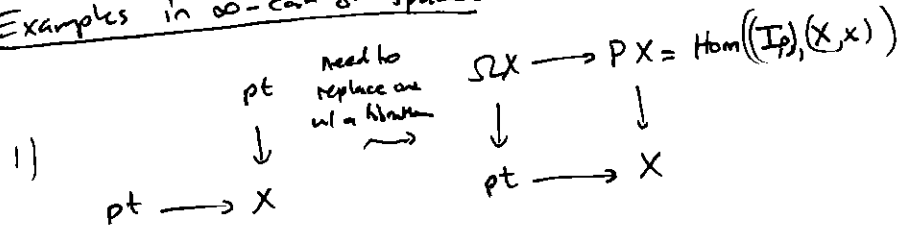
- Define slice in terms of join using adjunction:  $Fun^{(pt)}(\mathcal{D} \star I, \mathcal{C}) \cong Fun(\mathcal{D}, \mathcal{C}/F)$   
 $(1\text{-categorically: } F: I \rightarrow \mathcal{C}, \text{ slice cat } \mathcal{C}/F \text{ has objs } \begin{matrix} x \in I \\ c \in \mathcal{C} \\ (x, c, f) \\ f: c \rightarrow F(x) \end{matrix})$

- Define terminal object:  $c \in \mathcal{C}$  terminal if  $\mathcal{C}/c \rightarrow \mathcal{C}$  is a trivial fibration

Prop: full subcat of  $\mathcal{C}$  spanned by

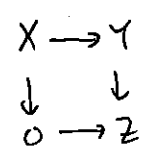
- $\lim F =$  terminal object of  $\mathcal{C}/F$

Examples in  $\infty$ -cat of spaces



Def: An  $\infty$ -category is pointed if it has an object that is both initial and terminal. (called 0)

Def: A triangle is a diagram



It is exact if it's a pullback, coexact if it's a pushout.

Def: Let  $f: X \rightarrow Y$ , map in  $\infty$ -cat.

A kernel to  $f$  is an exact triangle

$$\begin{array}{ccc} K & \rightarrow & X \\ \downarrow & & \downarrow f \\ 0 & \rightarrow & Y \end{array}$$

A cokernel to  $f$  is a coexact triangle

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ 0 & \rightarrow & K \end{array}$$

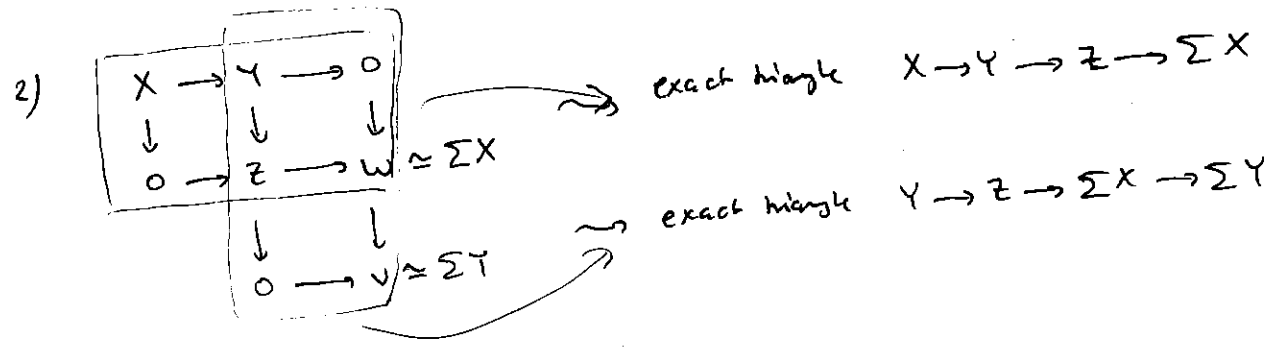
Def: An  $\infty$ -category is stable if

- it's pointed
- kernels and cokernels exist
- exact  $\Leftrightarrow$  coexact for triangles

Rmk: No extra structure! (as triangulated cat's have)

Fact:  $\mathcal{C}^{\text{stable}}$   $\infty$ -cat  $\Rightarrow$   $\text{ho } \mathcal{C}$  triangulated. Let's briefly discuss this:

1)  $\text{coker}(X \rightarrow 0) = \Sigma X$  : 
$$\begin{array}{ccc} X & \rightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \rightarrow & \Sigma X \end{array}$$





Def: A t-structure on a stable  $\infty$ -cat  $\mathcal{C}$  is a t-structure on  $\text{ho } \mathcal{C}$ .

$\Rightarrow \infty$ -subcat  $\mathcal{C}_{\geq n}, \mathcal{C}_{\leq n}$

$\mathcal{C}^{\heartsuit} := \mathcal{C}_{\geq 0} \cap \mathcal{C}_{\leq 0}$  is equivalent to  $N(\text{ho } \mathcal{C}^{\heartsuit})$

Let  $\mathcal{A}$  abelian category w/ enough projectives

Let  $\mathcal{C}$  stable  $\infty$ -category w/ t-structure.

$$\text{REx}(\mathcal{A}, \mathcal{C}^{\heartsuit}) \simeq \text{Fun}^{\text{R.t-Ex}}(\mathcal{D}^{-}(\mathcal{A}), \mathcal{C})^{(*)}$$

(\*) means: proj obj in  $\mathcal{A}$  are mapped to  $\mathcal{C}^{\heartsuit}$

Ex:  $\mathcal{A} = \mathcal{C}^{\heartsuit} \Rightarrow \mathcal{D}^{-}(\mathcal{C}^{\heartsuit}) \rightarrow \mathcal{C}$  realization

Ex:  $\mathcal{A} = (\text{Ab}), \mathcal{C} = (\text{Spectra}) \Rightarrow \mathcal{D}^{-}(\mathcal{A}) \rightarrow \mathcal{C}$ .  
 $\mathcal{C}^{\heartsuit}$

### Filtered objects

Def: A filtered obj in  $\mathcal{C}$  is a functor  $N(\mathbb{Z}) \rightarrow \mathcal{C}$ .

$$\begin{array}{ccc} X(n) & \rightarrow & X(n+1) \\ \downarrow & & \downarrow \\ 0 & \rightarrow & Y(n+1) = \text{coker } X(n) \rightarrow X(n+1) \end{array}$$

