DAG-seminar #14

K. Moi 1/6 - 2018

Quasi-categories as infinity categories

Idea of $(\infty, 1)$-categories:

An $(\infty,1)$-category $\mathcal{C}$ should be the following data:

- a set ob $\mathcal{C}$ of objects
- $\forall x, y \in \mathrm{ob}\,\mathcal{C}$, a set of morphisms $x \xrightarrow{f} y$
  and for $X \xrightarrow{f} Y \xrightarrow{g} Z$, a non-empty set of compositions $X \xrightarrow{h} Z$.
- Given compositions $h, h' : X \longrightarrow Z$, a non-empty set of invertible maps $h \xrightarrow{\sim} h'$ (2-morphisms)
- Given 2-morphisms ... etc.

Today: Look at implementation using quasi-categories.

nerve functor $N: \underset{\substack{\text{small cat's}\\ \text{\& functors}}}{Cat} \longrightarrow sSet$

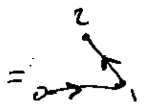
$$I \longmapsto N.I = \mathrm{Fun}([-], I)$$

$$N_n I = \{ \underset{x_0}{\cdot} \xrightarrow{f_1} \underset{x_1}{\cdot} \xrightarrow{f_2} \underset{x_2}{\cdot} \longrightarrow \cdots \xrightarrow{f_n} \underset{x_n}{\cdot} \}$$

N is fully faithful.

$X_\bullet \in sSet$ is iso to NI for some I iff $\forall\, \Lambda^n_i \longrightarrow X$, $\underset{0<i<n}{}$ $\exists$ unique lift to $\Delta^n \to X$

Def: $\Lambda^n_i = i^{th}$ horn of $\Delta^n = \Delta^n \setminus$ (top dim simplex and face opposite i)

Ex: $\Delta^2 =$ , $\Lambda^2_1 =$ 

Ex: $\Lambda^2_1 \longrightarrow NI \iff \underset{X}{\cdot} \xrightarrow{f} \underset{Y}{\cdot} \xrightarrow{g} \underset{Z}{\cdot} \iff \Delta^2 \longrightarrow NI$

The face opposite to 1 gives the unique composition $\underset{X}{\cdot} \xrightarrow{gf} \underset{Z}{\cdot}$

<u>Def</u>: A <u>quasicategory</u> is a sSet s.th. $\forall 0<i<n$, any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

"every inner horn has a filling"

has a lift.

<u>Ex</u>: The nerve of any ordinary category.

<u>Ex</u>: X any Kan complex (all horn fillings) (models ∞-cat.)

<u>Ex</u>: $Y \in \mathrm{Top}$, $\mathrm{Sing}(Y)$ Kan complex.

$$\begin{tikzcd} \mathrm{Grpds} \arrow[r, phantom, "\subseteq"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Cat} \arrow[d] \arrow[dr, "N"] & \\ \mathrm{Kan} \arrow[r, phantom, "\subseteq"] & \text{Quasi-cat} \arrow[r, phantom, "\subseteq"] & \mathrm{sSet} \end{tikzcd}$$

<u>Def</u>: A <u>functor</u> $F: \mathcal{C} \to \mathcal{D}$ b/w quasi-categories, is a map of simplicial sets.

A <u>natural tfm</u> $F \to G$ is a map of simplicial sets $\Delta^1 \times \mathcal{C} \xrightarrow{\Phi} \mathcal{D}$

s.th. $\Phi|_{0\times\mathcal{C}} = F$, $\Phi|_{1\times\mathcal{C}} = G$. (a simplicial homotopy)

Relation to simplicial categories

simplicial category := category enriched in sSets

set of objects $\mathrm{ob}\,\mathcal{C}$

$x, y \in \mathcal{C} \Rightarrow \mathrm{Map}(x,y) \in \mathrm{sSet}$

$X, Y, Z \in \mathcal{C} \Rightarrow \mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \xrightarrow{\circ} \mathrm{Map}(X,Z)$

Cat_Δ := category of simplicial categories

$\mathrm{Cat} \subset \mathrm{Cat}_\Delta$

"Building blocks"

$[n] = \{ \underset{0}{\cdot} \to \underset{1}{\cdot} \to \underset{2}{\cdot} \to \cdots \to \underset{n}{\cdot} \} \in Cat$

$\Delta^n \in sSet$

($\mathfrak{C}$ = Cordier '80s)

Def: Given J finite linearly ordered set, we define

$\mathfrak{C}[J]$ simplicial category with:

$ob\, \mathfrak{C}[J] = ob\, J$

$mor(i,j) = NP_{ij}$ $\quad P_{ij} = \{ \underset{i,j \in I}{I \subseteq J}, \ I \subseteq [i,j] \}$ poset ordered by inclusion

(all possible ways to go from i to j)

composition is nerve of:

$P_{ij} \times P_{jk} \xrightarrow{\cup} P_{ik}$

$\mathfrak{C}[-]$ is functorial in monotone maps $J \to J'$

Def: $\mathfrak{C}[\Delta^n] := \mathfrak{C}[0<1<\cdots<n]$

$\Delta \longrightarrow Cat_\Delta$

$[n] \longmapsto \mathfrak{C}[\Delta^n]$

Def: The coherent nerve of $\mathcal{D} \in Cat_\Delta$ is: (or Cordier nerve)

$N_\Delta \mathcal{D} := Fun_{Cat_\Delta}(\mathfrak{C}[-], \mathcal{D})$

$\mathcal{D} \in Cat_\Delta$

Description of coherent nerve $N_\Delta \mathcal{D}$:

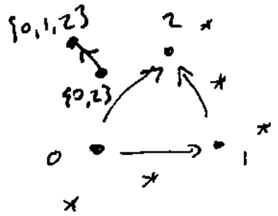
<u>0-simplices:</u> $\mathcal{C}[0] \longrightarrow \mathcal{D}$ so just objects of $\mathcal{D}$

$\mathcal{C}[0] = *$

<u>1-simplices:</u> $\mathcal{C}[\Delta^1] \longrightarrow \mathcal{D}$ so 0-simplices in mapping sets

$\mathcal{C}[\Delta^1] = \bullet \xrightarrow{*} \bullet$

<u>2-simplices</u> $\mathcal{C}[\Delta^2] \longrightarrow \mathcal{D}$ so two composable maps



$$\begin{tikzcd} & Z \\ X \arrow[ur, "h"] \arrow[ur, bend left=60, "gf"] \arrow[r, "f"'] & Y \arrow[u, "g"'] \end{tikzcd}$$

$h \overset{\alpha}{\Rightarrow} gf$ in $Map(X, Z)$

Composition in $\mathcal{D}$ is unique. The nerve $N_\Delta \mathcal{D}$ forgets this uniqueness by "adding all possible composites".

$$\begin{tikzcd} \Delta \arrow[r, "\mathcal{C}[-]"] \arrow[d, "\text{Yoneda}"'] & Cat_\Delta \arrow[dl, shift left, "N_\Delta"] \\ sSet \arrow[ur, shift left, "\mathcal{C}[-]"] & \end{tikzcd}$$

$[n] \mapsto \Delta^n$

cocomplete

left Kan extension

adjunction

$$X_\bullet = \varinjlim_{\Delta^n \to X_\bullet} \Delta^n \qquad \mathcal{C}[X_\bullet] = \varinjlim_{\Delta^n \to X_\bullet} \mathcal{C}[\Delta^n]$$

<u>Prop</u>: Let $\mathcal{D} \in Cat_\Delta$ such that all mapping sSets are Kan then $N_\Delta \mathcal{D}$ is a quasi-category

<u>Ex</u>: Take $Kan \subseteq sSet$ with enrichment:

$X, Y \in sSet$

$Map(X,Y)_n = Hom_{sSet}(X \times \Delta^n, Y)$

If Y Kan, then $Map(X,Y)$ Kan.

<u>Def</u>: $\mathcal{S} := N_\Delta Kan$

the ∞-category of spaces.

<u>Ex</u>: M simplicial model category (e.g. sSet)

Then M_{cf} (cofibrant & fibrant obj) is enriched in Kan complexes so $N_\Delta M_{cf}$ is a quasi-category.

Def: 1) $\mathcal{D} \in Cat_\Delta$. The homotopy category $h\mathcal{D}$ has:

- objects $=$ ob $\mathcal{D}$
- $Hom_{h\mathcal{D}}(X,Y) = \pi_0 Map_{\mathcal{D}}(X,Y)$ and composition is $\pi_0(\text{comp})$

2) A functor $F: \mathcal{C} \to \mathcal{D}$ in Cat_Δ is a Dwyer-Kan equivalence if $hF: h\mathcal{C} \to h\mathcal{D}$ is an equivalence of categories (or merely ess surj.) and $\forall x,y \in \mathcal{C}$, $F: Map_{\mathcal{C}}(x,y) \to Map_{\mathcal{D}}(Fx, Fy)$ weak equiv

3) A map $f: X \to Y$ of sSets is a categorical equivalence if $\mathfrak{C}[f]: \mathfrak{C}[X] \to \mathfrak{C}[Y]$ is a DK-equiv.

4) For $S \in sSet$, we define $hS = h\mathfrak{C}[S]$.

Thm (Joyal) There is a (combinatorial) model structure on sSets w/

- cofib $=$ levelwise inj maps
- weq $=$ categorical equivalence
- fibrant obj $=$ quasi-categories

Rmk: • All objects are cofibrant. so $(sSets)_{cf}$ = quasi-cat

$\Rightarrow$ ~~quasi~~ a categorical equiv b/w quasi-cat has a homotopy inverse.

• Julia Bergner put a Model Structure on Cat_Δ s.th. $sSet \underset{N_\Delta}{\overset{\mathfrak{C}[-]}{\rightleftarrows}} Cat_\Delta$ Quillen adjunction

$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$ $\quad$ $\mathcal{C}$ quasi-category

$$\Lambda^2_1 \overset{i}{\subseteq} \Delta^2 \qquad Map(\Delta^2, \mathcal{C}) \xrightarrow{i^*} Map(\Lambda^2_1, \mathcal{C})$$

<u>Fact</u>: i^* is trivial Kan fibration (that is fibers are contractible Kan cplxes)