DAG - seminar #13

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Artin's axioms

(0) stack
(1) limit preserving
(2) repr. diagonal
(3) exist formally versal def
(4) effectivity
(5) openness of versality

Reference: Hall "Openness of versality via coherent functors"

Deformation problem

$$\alpha = \left(\begin{tikzcd} X \arrow[r, hook, dashed] \arrow[d, "\pi"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, dotted] \\ S \arrow[r, hook, "I"] & S' \end{tikzcd} \right) \quad \pi \text{ flat}$$

Lemma: (1) $\exists$ solution iff obs class $o(\alpha) \in \mathrm{Ext}^2(\mathbb{L}_{X/S}, \pi^* I)$

(2) if $o(\alpha) = 0$, the set of solutions is an $\mathrm{Ext}^1(\mathbb{L}_{X/S}, \pi^* I)$ - torsor

(3) given a solution X', the set of automorphisms of X' (restricting to the identity on X) is iso to $\mathrm{Ext}^0(\mathbb{L}_{X/S}, \pi^* I)$

We will define groups: (for a stack $\mathfrak{X}$)

$\mathrm{Obs}_{\mathfrak{X}}(S, I)$
$\mathrm{Def}_{\mathfrak{X}}(S, I)$
$\mathrm{Aut}_{\mathfrak{X}}(S, I)$

with similar properties and also $\mathrm{Exal}_{\mathfrak{X}}(S, I)$. Then:

(openness of versality)

$\Updownarrow$

($\mathrm{Exal}_{\mathfrak{X}}(S, I)$ is finitely generated)

$\Uparrow$

($\mathrm{Def}_{\mathfrak{X}}(S, I)$ and $\mathrm{Obs}_{\mathfrak{X}}(S, I)$ are coherent)

Extensions

$\mathcal{X}$ CFG over scheme S. $\qquad I \in QCoh(T)$

Let T be a scheme and $T \xrightarrow{a} \mathcal{X}$ a morphism. An $\mathcal{X}$-extension of T by I is a diagram

$$\begin{tikzcd} T \arrow[rr, hook, "i"] \arrow[dr, "a"'] & & T' \arrow[dl, "a'"] \\ & \mathcal{X} & \end{tikzcd}$$

where i closed immersion w/ ideal I, $I^2=0$.

A morphism of $\mathcal{X}$-extensions of T by I is a morphism over $\mathcal{X}$

$$\begin{tikzcd} T \arrow[r, hook] \arrow[dr, hook] & T'' \arrow[dr] & \\ & T' \arrow[u] \arrow[r] & \mathcal{X} \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} 0 \arrow[r] & I \arrow[r] & \mathcal{O}_{T'} \arrow[r] & \mathcal{O}_T \arrow[r] & 0 \\ 0 \arrow[r] & I \arrow[r] \arrow[u, equal] & \mathcal{O}_{T''} \arrow[r] \arrow[u] & \mathcal{O}_T \arrow[r] \arrow[u, equal] & 0 \end{tikzcd}$$

So $T' \to T''$ is an iso.

This gives a category $\underline{\mathrm{Exal}}_{\mathcal{X}}(T, I)$.

and more generally a fibered category $\underline{\mathrm{Exal}}_{\mathcal{X}}(T)^{op} \to QCoh(T)$

$\mathcal{X}$-ext of T by $J \longmapsto J$

Let $\mathrm{Exal}_{\mathcal{X}}(T, I)$ be the set of iso-classes of $\underline{\mathrm{Exal}}_{\mathcal{X}}(T, I)$.

Fact: If $\mathcal{X}$ satisfies homogeneity (cond (H) of last time), then we get an additive functor:

$$\mathrm{Exal}_{\mathcal{X}}(T, -) : QCoh(T) \to (Ab)$$

If $T = \mathrm{Spec}\, A$, then the functor $\mathrm{Mod}(A) \to (Ab)$ lifts to $\mathrm{Mod}(A) \to \mathrm{Mod}(A)$.

Def: Let $F : \mathrm{Mod}\, A \to (Ab)$ be additive. Then F is:

(1) finitely generated if $\exists G \in \mathrm{Mod}\, A$ and $g \in F(G)$ s.th. $\forall M \in \mathrm{Mod}\, A$

$$\begin{aligned} \mathrm{Hom}(G, M) &\longrightarrow F(M) \\ f &\longmapsto f_* g \end{aligned} \quad \text{is surjective}$$

(2) coherent if $\exists (G \xrightarrow{\varphi} G') \in \mathrm{Mod}\, A$ and $g \in F(G)$ s.th. $\forall M \in \mathrm{Mod}\, A$

$$\mathrm{Hom}(G', M) \xrightarrow{\varphi^*} \mathrm{Hom}(G, M) \longrightarrow F(M) \longrightarrow 0$$

is exact.

(3) <u>half-exact</u> if $F(\text{s.e.s})$ is half-exact

$$0 \to M' \to M \to M'' \to 0 \quad \Rightarrow \quad F(M') \to F(M) \to F(M'') \text{ exact}$$
exact.

<u>Example</u>: $C^\bullet$ complex of A-modules, then

$$F : M \longmapsto H^i(\mathrm{Hom}(C^\bullet, M))$$

is coherent: a presentation is given by:

$$\mathrm{Hom}(C^{i+1}, M) \longrightarrow \mathrm{Hom}(C^i/d(C^{i-1}), M) \longrightarrow F(M) \longrightarrow 0$$

In particular, $M \longmapsto \mathrm{Ext}^i_A(C^\bullet, M)$ is coherent.

<u>Def</u>: Given $F : \mathrm{Mod}(A) \to (Ab)$ and $A \to B$, we have

$$F_B : \mathrm{Mod}(B) \longrightarrow \mathrm{Mod}(A) \xrightarrow{F} (Ab)$$

$$\mathbb{V}(F) := \{ p \in \mathrm{Spec}\, A : F_{A_p} = 0 \}$$

<u>Facts</u>: (1) If F fin.gen and commutes w/ direct limits, then $\mathbb{V}(F)$ is open.

(2) (Ogus-Bergman's Nakayama lemma) F half-exact and bounded, then $F(\kappa(p)) = 0$
$\Rightarrow$ ~~$F(A_p) = 0$~~
$F_{A_p} = 0$

<u>Lemma</u>: (1) If $\mathrm{Exal}_{\mathcal{X}}(T, I) = 0$ for all $I \in QCoh(T)$
then $T \to \mathcal{X}$ is formally smooth

(2) If $T \to \mathcal{X}$ is formally versal at $t \in |T|$ then
$$\mathrm{Exal}_{\mathcal{X}}(T, \kappa(t)) = 0$$

So to prove formally versal at $t \Rightarrow$ formally smooth in a nbhd
we would like $\mathrm{Exal}_{\mathcal{X}}(T, -)$ to be finitely generated.

<u>Def</u>: Let $\mathcal{Y} \xrightarrow{\Psi} \mathcal{Z}$ be a morphism of CFG's over S. Let T be an S-scheme.

$\underline{\mathrm{Def}}_\Psi(T, I)$ is the category of diagrams:

<u>Obj</u>:
$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & \mathcal{Y} \arrow[d, "\Psi"] \\ T[I] \arrow[r] \arrow[ur, dashed] & \mathcal{Z} \end{tikzcd}$$

Let $\mathrm{Def}_\Psi(T, I)$ be the iso. classes of $\underline{\mathrm{Def}}_\Psi(T, I)$.

<u>Lemma</u> If $\mathcal{Y}, \mathcal{Z}$ homogeneous, $\mathrm{Def}_\Psi(T, -) : \mathrm{QCoh}(T) \to \mathrm{Ab}$ additive.

and there's an exact sequence:

$$\mathrm{Def}_\Psi(T, I) \longrightarrow \mathrm{Exal}_{\mathcal{Y}}(T, I) \longrightarrow \mathrm{Exal}_{\mathcal{Z}}(T, I)$$

<u>Lemma</u> (5-lemma) $F_1 \to F_2 \to F_3 \to F_4 \to F_5$ exact seq of functors, then

(1) F_2, F_4 f.g, F_5 coh $\Rightarrow$ F_3 f.g.

(2) F_1, F_2 f.g, F_4, F_5 coh $\Rightarrow$ F_3 coh

$$\begin{array}{ccccccc} & & & & \text{coherent} & & \\ & & & & \mathrm{Ext}^1(\mathbb{L}_{T/S}, -) & & \\ & & & & \| & & \\ \mathrm{Def}_{\mathcal{X}/S}(T,-) & \to & \mathrm{Exal}_{\mathcal{X}}(T,-) & \xrightarrow{\alpha} & \mathrm{Exal}_S(T,-) & \longrightarrow & \mathrm{coker}(\alpha) \\ & & & & & & \| \\ & & & & & & \mathrm{Obs}_{\mathcal{X}}(T,-) \end{array}$$

Want to apply to the above.

minimal obs theory

<u>Example: Quot functor</u>

Let $X \xrightarrow{f} S$ morphism of schemes (separated + loc of fin pres)

and $\mathcal{F} \in \mathrm{QCoh}(X)$ flat over S.

$$Q = \mathrm{Quot}_{\mathcal{F}/X/S} : (\mathrm{Sch}/S)^{op} \longrightarrow (\mathrm{Set})$$

$$T \longmapsto \left\{ \mathcal{F}_T \twoheadrightarrow \mathcal{E} \in \mathrm{QCoh}(X_T) \text{ s.th } \mathcal{E} \text{ f. pres + flat } /T \text{ and } \mathrm{Supp}(\mathcal{E}) \to T \text{ proper} \right\} / \sim$$

Let $0 \to K \to \mathcal{F}_T \to \mathcal{E} \to 0$ be an element $x \in Q(T)$

and let $T \overset{J}{\hookrightarrow} T'$ be an element of $\mathrm{Exal}_S(T, J)$

We have a s.e.s.

$$0 \to \underset{\substack{\parallel \\ J\mathcal{F}_T}}{f_T^* J \otimes \mathcal{F}_T} \to \mathcal{F}_{T'} \to \mathcal{F}_T \to 0$$

and a s.e.s.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
0 \arrow[r] & f_T^* J \otimes K \arrow[r] & f_T^* J \otimes \mathcal{F}_T \arrow[r] \arrow[d, "i"] & f_T^* J \otimes \mathcal{E} \arrow[r] & 0 \\
 & & \mathcal{F}_{T'} \arrow[d, "\pi"] & & \\
0 \arrow[r] & K \arrow[r] & \mathcal{F}_T \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

gives $0 \to f^* J \otimes \mathcal{E} \to \pi^{-1}(K) / i(f_T^* J \otimes K) \to K \to 0$

i.e., an element of $\mathrm{Ext}^1_{\mathcal{O}_{X_T}}(K, f^* J \otimes \mathcal{E})$

and if this element is zero, get extension $0 \to f_T^* J \otimes K \to K' \to K \to 0$ fitting in the left of above diagram.

This gives $\mathrm{Exal}_{\cancel{Q}}(T, I) \to \mathrm{Exal}_S(T, I) \to \mathrm{Ext}^1_{\mathcal{O}_{X_T}}(K, f^* J \otimes \mathcal{E})$