DAG-seminar #12

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Outline

(1) Artin's axioms

(2) Examples of failures

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(3) Deformation/obs theory $\Rightarrow$ openness of versality

$S = \operatorname{Spec} k$ (more generally excellent scheme)

$\mathfrak{X}/S$ category fibred in groupoids. To prove algebraicity, we need to find a smooth atlas. Done through the following steps:

(i) existence of formally versal deformation

(ii) algebraization —— " ——

(iii) openness of versality

(iv) formally versal $\Leftrightarrow$ formally smooth

(i)
$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d] & \mathfrak{X} \\ \operatorname{Spec} R \arrow[ur, "\exists"] & \end{tikzcd}$$

R complete local ring

(ii)
$$\begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d] & \mathfrak{X} & \\ \operatorname{Spec} R \arrow[ur] \arrow[rr] & & U \arrow[ul, "p"'] \end{tikzcd}$$

U f.t./S

(iii) p is formally versal in a nbhd

(iv) p is smooth.

Def: Let $F : (Sch/S)^{op} \to (Sets)$ be a functor and $\varphi \in F(T)$.

Then $\varphi : T \to F$ is formally versal in $t \in |T|$ if $\forall$ diagrams

$$\begin{tikzcd} \operatorname{Spec} \kappa(t) \arrow[r, hook] \arrow[rr, bend left, "t"] & \operatorname{Spec} B/I \arrow[r] \arrow[d, hook] & T \arrow[d, "\varphi"] \\ & \operatorname{Spec} B \arrow[r] \arrow[ur, dashed, "\exists"] & F \end{tikzcd}$$

w/ B local artinian w/ residue field $\kappa(t)$, $I^2 = 0$

$\exists$ lift.

Artin's axioms

$\mathcal{X}/S$ category fibred in groupoids (CFG). Then $\mathcal{X}$ is an alg stack locfinite pres $/S$ iff S excellent.

(0) (stack) $\mathcal{X}$ is a stack in the étale top.

(1) (limit preserving) $\forall$ inverse systems of affine schemes $\{T_\lambda\}$: $\operatorname{colim}_\lambda \mathcal{X}(T_\lambda) \longrightarrow \mathcal{X}(\lim_\lambda T_\lambda)$

is an equivalence of groupoids.

(2) (repr diagonal) $\Delta : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ repr. by alg spaces.

(3) (existence of form. versal def.)

(4) (effectivity) $\forall$ complete local Noetherian k-alg (R, m)

$$\mathcal{X}(\operatorname{Spec} R) \longrightarrow \lim_n \mathcal{X}(\operatorname{Spec} R/m^n) \text{ equivalence of grpds.}$$

(5) (openness of versality)

Homogeneity

$I^2 = 0$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B \arrow[d] \\ R \arrow[r] & B/I \end{tikzcd}$$

Cartesian diagram of rings $\left(A = B \times_{B/I} R\right)$

gives $\mathcal{X}(\operatorname{Spec} A) \xrightarrow{\Phi} \mathcal{X}(\operatorname{Spec} R) \times_{\operatorname{Spec}(B/I)} \operatorname{Spec}(B)$

$\mathcal{X}$ homogeneous if Φ equiv. of grpds.

Counter-examples

① $R_n = k[x,y]/(y(y-x^{2n}))$ $\quad X_n = \operatorname{Spec} R_n$

$X_n \longrightarrow X_{n+1}$

$(x,y) \longmapsto (x, x^2 y)$

$X := \operatorname{colim}_n X_n$

as a functor $(Sch)^{op} \to (Set)$

$\phi_1 : \operatorname{Spec} k[[x]] \longrightarrow X$

$\phi_1^{(n)} : \operatorname{Spec} k[[x]] \longrightarrow X_n$

$x \longmapsto (x, 0)$

$\phi_2^{(n)} : \quad x \longmapsto (x, x^{2n})$

Form fibre product

$$\begin{array}{ccc} W & \longrightarrow & \operatorname{Spec} k[[x]] \\ \downarrow & \square & \downarrow \phi_1 \times \phi_2 \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

$$W \cong \operatorname{colim}_n \underbrace{X_n \times_{X_n \times X_n} \operatorname{Spec} k[[x]]}$$

$$= \operatorname{Spec} \underbrace{R_n \otimes_{R_n \otimes R_n} k[[x]]}$$

$$= k[[x]]/(x^{4n})$$

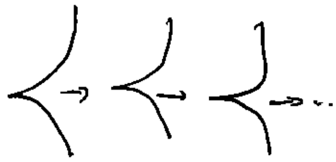
b/c $\quad 1 \otimes x^{2n} x^{2n} = 1 \otimes \phi_1(y - x^{2n}) \phi_2 (\ldots$

so $W \cong \operatorname{Spf} k[[x]]$ not algebraic space: <u>cond (2) fails</u>.

(2) $R_n = k[x,y]/(xy \prod_{i=1}^{n} (x - iy))$ or b) $R_n = k[x,y]/(y^2 - x^{2n+1})$

a)

$$\begin{array}{ccc} R_{n+1} & \longrightarrow & R_n \\ x & \longmapsto & x \\ y & \longmapsto & xy \end{array}$$



$X_n = \operatorname{Spec} R_n$, $\quad X = \operatorname{colim}_n X_n$

<u>Claim</u>: Cond (4) fails.

<u>pf</u>: want $\operatorname{colim}_n X_n(\operatorname{Spec} k[[x]]) \longrightarrow \lim_r \operatorname{colim}_n X_n(\operatorname{Spec} k[x]/(x^r))$ (not equiv) not <u>surj</u>.

For each r, $\exists n$ and $R_n \xrightarrow{\varphi_r} k[x]/(x^r)$ s.th.

$$\begin{array}{ccc} R_m & \xrightarrow{\varphi_{r+1}} & k[x]/(x^{r+1}) \\ \downarrow & & \downarrow \\ R_n & \xrightarrow{\varphi_r} & k[x]/(x^r) \end{array}$$

In a) Choose $R_n \longrightarrow k[x]/(x^r)$ $\quad r \leq n+2$

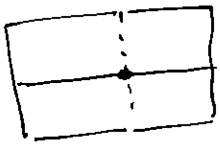
$y \longmapsto 2x$

In b) Choose $R_n \longrightarrow k[x]/(x^{r})$ $\quad r \leq 2n+1$

$y \longmapsto 0$

(3) $F : (Sch/k) \longrightarrow (Set)$

$$T \longmapsto \{ \varphi : T \longrightarrow \mathbb{A}^2_k : \exists\, T = T_1 \cup T_2 \text{ open cov.}$$
$$\varphi(T_1) \subseteq \{y=0\},\ \varphi(T_2) \subseteq \{x \neq 0\} \}$$



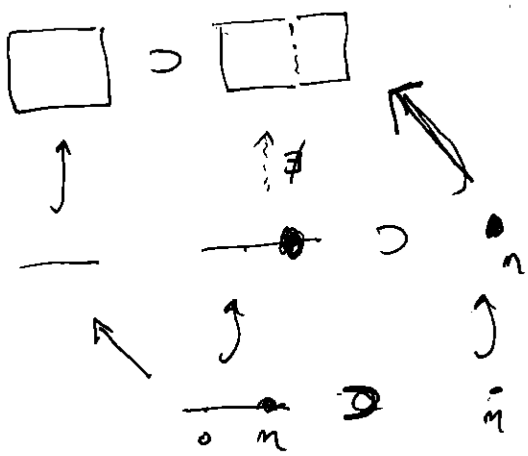
$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] & k((s))[\varepsilon]/(\varepsilon^2) \arrow[d, two heads] \\ k[[s]] \arrow[r] & k((s)) \end{tikzcd}$$

$A = k[[s]] + \varepsilon k((s))$

Claim: $F(A) \neq F(k[[s]]) \times_{F(k((s)))} F(k((s))[\varepsilon]/(\varepsilon^2))$

$$\begin{tikzcd} k[x,y] \arrow[r] \arrow[d] & k[x,x^{-1},y] \arrow[drr] & & \\ k[x] \arrow[drr] & & A \arrow[r] \arrow[d, two heads] & k((s))[\varepsilon]/(\varepsilon^2) \arrow[d, two heads] \\ & & k[[s]] \arrow[r] & k((s)) \end{tikzcd}$$

$x \mapsto s$, $y \mapsto \varepsilon$; $x \mapsto s$



But $\underset{\overset{\shortparallel}{T}}{\operatorname{Spec} A}$ has a unique closed point so $T = T_1$ or $T = T_2$.

But there is no lift $\operatorname{Spec} A \longrightarrow \operatorname{Spec} k[x, x^{-1}, y]$

or $\operatorname{Spec} A \longrightarrow \operatorname{Spec} k[x]$