# Bridgeland stability conditions

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§1 Definition of BSC

§2 BSC on surfaces

§3 Birat. geom of m.s of sheaves

Refs:

| | |
|---|---|
| Bridgeland | arXiv: 02/2237 |
| Macrí et al | arXiv: 1607.01262 |
| D-branes ... | arXiv: |
| Liu | arXiv: 1501.06397 |

## §1 Basic notions

$A \rightsquigarrow D(A) \cong K(A)[S^{-1}]$ $S = \{q\text{-is}\,o\}$

**translation functor** $(E[1])_i := E_{i+1}$

**triangles**: $E \xrightarrow{\varphi} F \to \mathrm{Cone}(E \xrightarrow{\varphi} F) \to E[1]$

$D^b(A) = \{E \in D(A) : H^i(E) = 0 \ \forall |i| \gg 0\}$

$X$ sm proj var$/\mathbb{C}$: $D(X) = D^b(\mathrm{Coh}\, X)$

**Def**: The heart of a t-structure on $D(X)$ is a full additive subcat $\mathcal{A} \subset D(X)$ s.th.

(1) $\forall i > j$ integers and $A, B \in \mathcal{A}$:

$$\mathrm{Hom}_{D(X)}(A[i], B[j]) = \mathrm{Hom}_{D(X)}(A, B[j-i]) = 0 \qquad \text{(no negative exts)}$$

(2) $\forall E \in D(X)$ $\exists$ integers $k_1 > \dots > k_n$ and $E_i \in D(X)$ and $A_i \in \mathcal{A}$ and coll. of triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[dl] \arrow[r] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& \mathrm{cone}(E_0 \to E_1) \arrow[ul, dashed, "{[1]}"] & & & & \mathrm{cone}(E_{m-1} \to E_m) \arrow[ul, dashed, "{[1]}"] &
\end{tikzcd}$$

where $\mathrm{cone}(E_i \to E_{i+1}) \cong A_i[k_i]$

**Thm** (BBD) The heart of a t-structure $\mathcal{A}$ is abelian.

**Rmk**: $K_0(\mathcal{A}) = K_0(D(X)) = K_0(X)$

<u>Def</u>: A slicing $\mathcal{P}$ of $D(X)$ is a collection of full subcategories

$P(\phi) \subset D(X)$, $\phi \in \mathbb{R}$ s.th.

① $P(\phi+1) = P(\phi)[1]$

② $\mathrm{Hom}_{D(X)}(P(\phi_1), P(\phi_2)) = 0 \quad \forall \phi_1 > \phi_2$

③ $\forall E \in D(X)$ $\exists$ real numbers $\phi_1 > \dots > \phi_m$ and $E_i \in D(X)$

s.th. $0 = E_0 \to E_1 \to \cdots \to E_m = E$

s.th. $\mathrm{Cone}(E_{i-1} \to E_i) = A_i \in P(\phi_i)$

<u>Def</u> (Bridgeland) A <u>Bridgeland stability condition</u> on $D(X)$ is a pair

$\sigma = (\mathcal{P}, Z)$ where

- $\mathcal{P}$ <u>slicing</u>
- $Z$ <u>central charge</u>: $Z : K_0(X) \xrightarrow{v} \underset{\text{lattice}}{\Lambda} \to \mathbb{C}$ additive hom

s.th.

① $\forall\, 0 \neq E \in P(\phi)$, $Z(E) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$

② $$c_\sigma := \inf\left\{ \frac{|Z(E)|}{\|v(E)\|} : \begin{array}{c} 0 \neq E \in P(\phi) \\ \phi \in \mathbb{R} \end{array} \right\} > 0$$

<u>Def</u>: $E \in P(\phi)$ is called semi-stable obj w/ phase $\phi$.

<u>Rmk</u>: Enough to give $\sigma = (\underset{\substack{\uparrow \\ \text{obj of phase } (0,1]}}{\mathcal{A}}, Z)$

$E \in \underset{\mathcal{A}}{\cancel{D(X)}}$ is $\sigma$-s.s. if $\forall\, 0 \neq \underset{\substack{\cap \\ \mathcal{A}}}{F} \subset E$, $\phi_\sigma(F) < \phi_\sigma(E)$

## Space of stability conditions

$\mathrm{Stab}(D(X),v) = \{\sigma\}$ comes w/ natural topology.

metric given by distance: $d(\sigma_\bullet, \sigma'_\bullet) = \inf\limits_{E\in D(X)} \left\{ \begin{array}{l} \phi'_{max}(E) - \phi_{max}(E) \\ \phi'_{min}(E) - \phi_{min}(E) \\ |Z'(E)| - |Z(E)| \end{array} \right\}$

Forgetful map
$$\begin{array}{ccc} \mathrm{Stab}(D(X),v) & \longrightarrow & \mathrm{Hom}(\Lambda, \mathbb{C}) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \sigma = (P, Z) & \longmapsto & Z \end{array}$$

is local homeo. So a manifold of dim $= \mathrm{rk}\,\Lambda$

$$\begin{array}{c} \mathrm{Aut}(D(X)) \circlearrowright \mathrm{Stab} \circlearrowleft \widetilde{GL}_2^+(\mathbb{R}) \\ \qquad\qquad\qquad\qquad\quad \cup \\ \qquad\qquad\qquad\qquad\quad \mathbb{C} \end{array}$$

Examples

- $C$ curve, $g(C) \geq 1$ (Bridgeland, Macri)

  $\mathrm{Stab}(D(C)) \cong \mathbb{H} \times \mathbb{C}$

  $\rotatebox{90}{$\in$}$

  $\sigma_0 = (\mathrm{Coh}(C), Z(E) := -\deg(E) + i\,\mathrm{rk}(E))$

- $C$ curve, $g(C) = 0$, $\mathrm{Stab}(D(C)) \cong \mathbb{C}^2$

- $X$ smooth proj surface $/\mathbb{C}$

  $\omega \in \mathrm{Amp}(X)$, $\beta \in NS^1_{\mathbb{R}}(X)$

$$Z_{\omega,\beta}(-) = -\int_X e^{-(\beta+i\omega)} ch(-)$$
$$= -\int_X \left(1, -(\beta+i\omega), \frac{\beta^2-\omega^2}{2} + i\beta\omega\right)\cdot(ch_0, ch_1, ch_2)$$
$$= -ch_2 + \frac{1}{2}ch_0(\omega^2-\beta^2) + ch_1\beta + i\underbrace{\omega(ch_1 - ch_0\beta)}_{= ch_0(\mu_\omega(E) - \omega\cdot\beta)}$$

**Goal**: Find a heart $\mathcal{A}$ s.th. $\mathrm{Im}(Z(A)) > 0$

or $= 0 \Rightarrow -\mathrm{Re}\, Z > 0$.

If $ch_0 > 0$: $\mu_\omega(E) = \dfrac{ch_1(E).\omega}{ch_0(E)}$

Def $E \in \mathrm{Coh}\, X$ is **$\mu_\omega$-(semi)stable** if $\forall F \subset E$ satisfy $\mu_\omega(F) \underset{(=)}{<} \mu_\omega(E)$

$\forall E$ exists **Harder-Narasimhan filtration** $0 = E_0 \subset \cdots \subset E_m = E$

s.th. $E_i / E_{i-1}$ is $\mu_\omega$-semistable.

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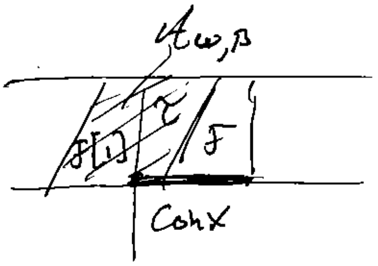
$\mathcal{T} := \{ E \in \mathrm{Coh}\, X : (E_i/E_{i-1}\ \mu_\omega\text{-s.s. factor has slope})\ \mu_\omega(E_i/E_{i-1}) > \omega\beta \}$

$\mathrm{Im}(Z(\mathcal{T})) > 0$

$\mathcal{F} := \{ E \in \mathrm{Coh}\, X : \mu_\omega(E_i/E_{i-1}) \leq \omega\beta \}$

$\mathrm{Im}(Z(\mathcal{F}))$



**Lem** (Bridgeland, Arcara-Bertram) $\sigma_{\omega,\beta} = (\mathcal{A}_{\omega,\beta}, Z_{\omega,\beta}) \in \mathrm{Stab}(D(X))$

$\mathcal{A}_{\omega,\beta} = \langle \mathcal{T}, \mathcal{F}[1] \rangle$

**key part of pf**: If $E = F[1]$ w/ $F \in \mathcal{F}_{\omega,\beta}$ and $\mu_\omega(F) = \omega\beta$, then $\mathrm{Re}(Z(F[1])) < 0$

Bogomolov-Gieseker ineq + Hodge index thm.

## § Application

$X$ smooth $\overset{\text{proj}}{\text{surface}}$ $/\mathbb{C}$

$\omega \in \mathrm{Amp}(X)$

$\alpha \in NS(X)_{\mathbb{Q}}$

$M := M_{(\alpha,\omega)}(ch)$ = $\alpha$-twisted $\omega$-Gieseker mod space of fixed top type $ch$.

$E \in \mathrm{Coh}\,X$ is $(\alpha,\omega)$-semistable if $\forall\, 0 \neq F \subsetneq E$

$$\frac{\chi(F \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{\text{leading coeff w.r.t. } m} \underset{\underset{\forall m >\!> 0}{\uparrow}}{\leq} \frac{\chi(E \otimes \alpha^{-1} \otimes \omega^{\otimes m})}{\text{leading coeff wrt } m}$$

<u>Rmk</u>: (i) If $\omega = tH$, $t>0$, then $(\alpha,\omega)$-(s)s $\Leftrightarrow$ $(\alpha, H)$-(s)s.

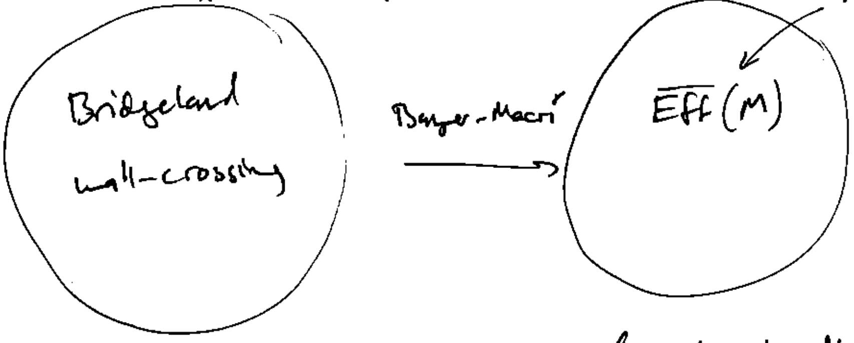
(ii) If $\alpha = \beta - \frac{1}{2}K_X$ and $t >\!> 0$, $\omega = tH$, then $M_{\sigma_{\omega,\beta}}(ch) \cong M_{(\alpha,\omega)}(ch)$

<u>More precisely</u>: $\sigma_{\omega,\beta}$-s.s. objects are sheaves and $(\alpha,\omega)$-s.s.

$(\alpha,\omega)$-ss sheaves are $\sigma_{\infty,\beta}$-ss obj in $\mathcal{A}_{\omega,\beta}$.

$M = M_{\sigma_{\omega,\beta}}(ch) = M_{(\alpha,\omega)}(ch)$

Indep. of $\sigma$ (existence of global Bayer-Macrì)



$\boxed{\text{Claim}}$ $NS(M_\sigma(ch)) \cong NS(M_{\sigma'}(ch))$

$X = K3$
$X = \mathbb{P}^2$ Li-Zhou
$\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{F}_1$

$\sigma \longmapsto l_\sigma$ line bundle on $M$

$l_\sigma$ defined via intersection numbers:

$$l_\sigma . C = \mathrm{Im} \frac{Z(\Phi_E(\mathcal{O}_C))}{-Z(ch)}$$

$C$ curve on $M$

$$\begin{tikzcd} & \mathcal{E} \\ & C \subset M \times X \arrow[dl] \arrow[dr] & \\ X & & C \subset M \end{tikzcd}$$

$\Phi_E : D(M) \longrightarrow D(X)$

$l_\sigma \in \mathrm{Nef}(M_\sigma(ch))$