I Definitions

II Triang. str. on $D(A)$

III Triang. cat. and duality

IV Witt-groups

I. **Def:** Let C be an additive category. A triangulation on C is:

(1) An additive automorphism
$$\Sigma : C \longrightarrow C \qquad (X[n] := \Sigma^n X)$$

(2) A collection Δ of "distinguished triangles"
$$X \to Y \to Z \to X[1]$$

s.t.

TR1:
- $X \xrightarrow{id} X \to 0 \to X[1] \in \Delta$
- $X \xrightarrow{f} Y \dashrightarrow Z \to X[1]$
- Δ closed under iso.

TR2: If $X \xrightarrow{u} Y \to Z \xrightarrow{w} X[1]$ in Δ

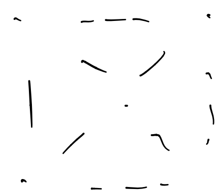
then so are
- $Y \to Z \to X[1] \xrightarrow{-u[1]} Y[1]$
- $Z[-1] \xrightarrow{-w[-1]} X \to Y \to Z$

TR3:
$$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & X[1] \\ f\downarrow & & \downarrow & & \vdots & & \downarrow f[1] \\ X' & \to & Y' & \to & Z' & \to & X'[1] \end{array}$$

TR4: Given $\begin{cases} \alpha: X \xrightarrow{u} Y \to Z' \to \\ \beta: Y \xrightarrow{v} Z \to X' \to \\ \gamma: X \xrightarrow{vu} Z \to Y' \to \end{cases}$

$\exists \quad \delta: Z' \to Y' \to X' \xrightarrow{p}$ s.t.

$$\begin{tikzcd} X' \arrow[d, "p"', "{[1]}"] \arrow[dr, "{[1]}"] & & Z \arrow[ll] \\ & Y \arrow[ur] \arrow[dl] & \\ Z' \arrow[rr, "{[1]}"] & & X \arrow[uu] \end{tikzcd}$$

&  ?

II. $\mathcal{D}(\mathcal{A})$ $\mathcal{A}$ - abelian

def. shift and cone as usual for $\mathcal{D}(\mathcal{A})$. Then dist. tr. are of the form $A \xrightarrow{f} B \to C(f) \to A[1]$.

Lemma: Let $A^\cdot \xrightarrow{f} B^\cdot \to (B/A)^\cdot$ be exact. Then the maps

$$\begin{array}{ccccc} A^n & \longrightarrow & B^n & \longrightarrow & B^n/A^n \\ \uparrow id & & \uparrow id & & \uparrow \varphi^n = (pr, 0) \\ A^n & \longrightarrow & B^n & \longrightarrow & B^n \oplus A^n \end{array}$$

are chain maps which gives a q.iso. and a homot. eq. if f is split.

Def: Let C be a tr. cat., W a class of morph. in C. A <u>Verdier localization</u> is an exact functor
$$\gamma : C \longrightarrow C[W^{-1}]^{\Delta}$$
which takes W to isomorphisms.

$$\begin{tikzcd}
C \arrow[r, "F"] \arrow[d, "\gamma"'] & D \\
C[W^{-1}]^{\Delta} \arrow[ur, dashed, "\exists ! \bar{F}"'] &
\end{tikzcd}$$

Fact: The localization $K(\mathcal{A}) \longrightarrow D(\mathcal{A})$ is a Verdier localization.

⋮

III Duality:

Def: Let C be a triang. cat. Then a duality on C is a triple (D, δ, ω),

(1) $\delta = \pm 1$

(2) $D : C^{op} \to C$ δ-exact:

* $\Sigma D = D \Sigma^{-1}$
* $A \to B \to C \xrightarrow{w} A[1]$ dist. $\Rightarrow$ $D(C) \to D(B) \to D(A) \xrightarrow{\delta \cdot D(w)[1]} (D(C)[1])$ dist.

(3) ω is an iso. $\mathbb{1} \to \mathcal{D}^2$ s.t. $\forall A$:

$$\begin{tikzcd}
\mathcal{D}A \arrow[dr, equal] \arrow[rr, phantom, shift right=4ex, "\circlearrowleft"] & & \mathcal{D}^3 A \arrow[ll, "\mathcal{D}(\omega_A)"'] \\
& \mathcal{D}A \arrow[ur, "\omega_{\mathcal{D}A}"'] &
\end{tikzcd}$$

(4) $\Sigma(\omega_A) = \omega_{\Sigma A}$.

Ex: Define
$$\mathcal{D}^{(i)} := \Sigma^i \mathcal{D}$$
$$\delta^{(i)} := (-1)^i \delta$$
$$\omega^{(i)} := \delta^i (-1)^{\frac{i(i+1)}{2}} \omega.$$
Then $(\mathcal{D}^{(i)}, \delta^{(i)}, \omega^{(i)})$ is called the i-shifted duality.

Ex: Let X be a regular scheme. Then $\mathcal{F}^\vee$ gives a duality on $\mathcal{D}^b(\mathrm{Coh}(X))$.

Def: A map $\varphi : A \to \mathcal{D}^{(i)} A$ is <u>i-symmetric</u> if

$$\begin{tikzcd}
A \arrow[r, "\varphi"] \arrow[d, "\omega_A^{(i)}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}^{(i)} A \arrow[d, equal] \\
(\mathcal{D}^{(i)})^2(A) \arrow[r] & \mathcal{D}^{(i)} A
\end{tikzcd}$$

• Let $Sym^i(C)$ be the monoid of <u>isometry classes</u> of i-sym. pairs with addition given by the <u>orth. sum</u>.

<u>Rmk</u>: $Sym^i(C,D,\delta,\omega) = Sym^0(C,D^{(i)},\delta^{(i)},\omega^{(i)})$

<u>Def</u>: (A,φ), (B,ψ) are <u>isometric</u> if

$$\begin{tikzcd} A \arrow[r, "\cong"] \arrow[r, "f"'] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[d, "\psi"] \\ D^{(i)}A & D^{(i)}B \arrow[l, "Df"'] \end{tikzcd}$$

<u>Def</u>: $(A,\varphi)\oplus(B,\psi) := (A\oplus B,\ D^{(i)}\varphi + D^{(i)}\psi)$

<u>Theorem</u>: Suppose C is uniquely 2-divisible. Let (A,φ) be an i-sym. pair. Then $\exists$ dist. tr.

$$(*)\quad \begin{tikzcd} A \arrow[r, "\varphi"] & D^{(i)}A \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowleft"] & C \arrow[r, "\beta"] \arrow[d, dashed, "\cong"', "\psi"] & A[1] \\ & D^{(i)}A \arrow[r, "\delta^{(i+1)} D^{(i+1)}\beta"'] & D^{(i+1)}C & \end{tikzcd}$$

(C,ψ) is i-sym. and unique up to isometry.

we get a well-def. map

$$\begin{aligned} \mathrm{Sym}^i(C) &\xrightarrow{d^i} \mathrm{Sym}^{i+1}(C) \\ (A, \varphi) &\longmapsto (C, \psi) \end{aligned}$$

$d^{i+1} d^i = 0$, d_i resp. $\oplus$

Suppose we are given (*)

$$C \xrightarrow[\cong]{\psi} D^{(i+1)} C \longrightarrow 0$$

<u>Def:</u> The Witt group of C

$$W^i(C) := \ker d_i / \operatorname{im} d_{i-1}$$

<u>Rmk:</u> This is a group: If $(A, \varphi) \in \ker d^i$ then $A \xrightarrow[\cong]{\varphi} D^{(i)} A \longrightarrow 0$ so $(A, -\varphi)$ is the inverse of (A, φ).

<u>Ex:</u> How does these appear for $C = D(\mathrm{Coh}(X))$ X regular ?