III. Properties:

k comm. ring (not nec. a field)

Thm: (Flat base change)
Let $k \to k'$ be flat, A a k-alg., and $B = A \otimes_k k'$. Then

$$B \otimes_A L_{A/k} \simeq L_{B/k'}$$

---

pf: Let $P_\bullet \to A$ be a cofib. resolution. By flatness $k' \otimes_k P_\bullet \to k' \otimes_k A = B$ is a weak equiv. Furthermore $k' \otimes_k P_\bullet$ is cofibrant. So it suffices to show that

$$B \otimes_A \Omega_{A/k} \cong \Omega_{B/k'}$$

which is true even without flatness. Π

Prop: If $U \subseteq k$ is a mult. subset,

$$k \longrightarrow R = U^{-1}k.$$

Then $L_{R/k} \simeq 0$

---

pf: $L_{R/k}$ is a simpl. R-mod. so

$$L_{R/k} \otimes_k R \cong L_{R/k}.$$

By flat BC

$$L_{R/k} = L_{R/k} \otimes_k R \simeq L_{R \otimes_k R/R} \cong L_{R/R} \simeq 0 \qquad \Pi$$

Thm: Given $k \to A \to B$, the seq.
$$B \otimes_A L_{A/k} \longrightarrow L_{B/k} \longrightarrow L_{B/A}$$
is a cofib. seq. (dist. triangle).

pf: Choose a free resolution
$$P_\bullet \twoheadrightarrow A.$$

Then

$$\begin{tikzcd}
P_\bullet \arrow[r, two heads, "\sim"] \arrow[d, hook] & A \arrow[d] \\
Q_\bullet \arrow[r, two heads, "\sim"] & B
\end{tikzcd}$$
for some $Q_\bullet$

Can be chosen as a free map. (for ex. bar res. of $P_\bullet / B$)

Then we have a SES

$$\Omega_{P_\bullet/k} \otimes_{P_\bullet} Q_\bullet \to \Omega_{Q_\bullet/k} \to \Omega_{Q_\bullet/P_\bullet} \to 0 \quad \text{(levelwise)}$$

Apply $- \otimes_{Q_\bullet} B$:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_{P_\bullet/k} \otimes_{P_\bullet} B \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \Omega_{Q_\bullet/k} \otimes_{Q_\bullet} B \arrow[r] \arrow[d, phantom, "="] & \Omega_{Q_\bullet/P_\bullet} \otimes_{Q_\bullet} B \arrow[r] \arrow[d, phantom, "\cong\,?"] & 0 \\
& L_{A/k} \otimes_A B & L_{B/k} & L_{B/A} &
\end{tikzcd}$$

(Because $P_\bullet \hookrightarrow Q_\bullet$ has levelwise retraction)

$$\begin{tikzcd}
P_\bullet \arrow[r] \arrow[d, hook] & A \arrow[d] \arrow[ddr, bend left] & \\
Q_\bullet \arrow[r, "\simeq"] \arrow[drr, bend right, two heads, "\sim"'] & Q_\bullet \otimes_{P_\bullet} A \arrow[dr, "\sim"] & \\
& & B
\end{tikzcd}$$

By 2 out of 3: weak equiv.

So $L_{B/A} \simeq \Omega_{A \otimes_{P_\bullet} Q_\bullet / A} \otimes_{A \otimes_{P_\bullet} Q_\bullet} B \simeq \Omega_{Q_\bullet/P_\bullet} \otimes_{Q_\bullet} B$

$\underbrace{\Omega_{A \otimes_{P_\bullet} Q_\bullet / A}}_{\text{Base change}} \simeq \Omega_{Q_\bullet/P_\bullet} \otimes_{Q_\bullet} A \otimes_{P_\bullet} Q_\bullet$

$\square$

Cor: $k \longrightarrow R$, $U \subseteq R$ mult. subset
Then
$$L_{U^{-1}R/k} \simeq U^{-1} L_{R/k}$$

Let $r \in k$ and consider
$$k \longrightarrow k[x] \xrightarrow{x \mapsto r} k$$

$$\rightsquigarrow \underset{\displaystyle \begin{array}{c} \wr\wr \\ k\langle dx\rangle \end{array}}{k \otimes_{k[x]} L_{k[x]/k}} \longrightarrow \underset{\displaystyle \begin{array}{c} \wr\wr \\ 0 \end{array}}{L_{k/k}} \longrightarrow L_{k/k[x]} \qquad \text{cofib. seq.}$$

$$\Longrightarrow \quad L_{k/k[x]} \simeq k\langle dx\rangle[1].$$

Now consider
$$\begin{tikzcd}
k[x] \arrow[r, "x \mapsto 0"] \arrow[d, "x \mapsto r"'] \arrow[dr, phantom, "\ulcorner", very near start] & k \arrow[d] \\
k \arrow[r] & k/(r)
\end{tikzcd}$$

Take a resolution:
$$\begin{tikzcd}
k[x] \arrow[r, hook] \arrow[rr, bend right, "x \mapsto 0"'] \arrow[d, "x \mapsto r"'] & B_\bullet \arrow[r, "\sim"] \arrow[d] & k \arrow[d] \\
k \arrow[r, hook] \arrow[rr, bend right] & A_\bullet \arrow[r] & k/(r)
\end{tikzcd}$$

Then $\Omega_{A_\bullet/k} \otimes_{A_\bullet} k/(r) \simeq \Omega_{B_\bullet/k[x]} \otimes_{B_\bullet} A_\bullet \otimes_{A_\bullet} k/(r)$

$\simeq \Omega_{B_\bullet/k[x]} \otimes_{B_\bullet} k \otimes_k k/(r)$

$\simeq L_{k/k[x]} \otimes_k k/(r)$

$= \Sigma\, k/(r)$ (Σ suspension)

If $A_\bullet \longrightarrow k/(r)$ was a w-eq

we'd have $L_{k/(r)/k} \simeq \Sigma\, k/(r)$.

Check this on normalised ch. compl.

Take $NB_\bullet = \{ k[x] \xrightarrow{x} k[x] \}$

then $NA_\bullet = \{ k \xrightarrow{r} k \}$. This is a k-resolution of $k/(r)$ iff r is not a zero divisor.

Thus $L_{k/k[x]} \simeq \Sigma\, k/(r)$ if r is not a zero divisor.

Cor: If $k \longrightarrow R$ is locally a complete intersection (reg. seq.) then flat dim $(L_{R/k}) \leq 1$.

Converse true (Noeth.) by Lichtenbaum-Schlessinger.

IV <u>Visitas:</u> k Noeth. R finite type.

$$L_{R/k} \simeq 0 \iff R/k \text{ étale.}$$
$$\text{f.d } L_{R/k} = 0 \iff R/k \text{ smooth}$$

<u>Conjecture:</u> (Quillen) If the flat dim of $L_{R/k}$ is finite then

$$\text{flat dim}(L_{R/k}) \leq 2$$

proven in a special case by Avramov.