

II. Properties: k comm. ring
(not nec. a field)

Thm: (Flat base change)

Let $k \rightarrow k'$ be flat, A a k -alg., and
 $B = A \otimes_k k'$. Then

$$B \otimes_A L_{A/k} \cong L_{B/k'}$$

pf: Let $P. \rightarrow A$ be a colts. resolution.

By flatness $k' \otimes_k P. \rightarrow k' \otimes_k A = B$ is a
weak equiv. Furthermore $k' \otimes_k P.$ is coltsmb.

So it suffices to show that

$$B \otimes_A \Omega_{A/k} \cong \Omega_{B/k'}$$

which is true even without flatness. \square

Prop: If $U \subseteq k$ is a mult. subset,
 $k \rightarrow R = U^{-1}k$.

Then $L_{R/k} \cong 0$

pf: $L_{R/k}$ is a simpl. R -mod. so

$$L_{R/k} \otimes_k R \cong L_{R/k}$$

By flat BC

$$L_{R/k} = L_{R/k} \otimes_k R \cong L_{R \otimes_k R/R} \cong L_{R/R} \cong 0 \quad \square$$

Thm: Given $k \rightarrow A \rightarrow B$, the seq.
 $B \otimes_A L_{A/k} \rightarrow L_{B/k} \rightarrow L_{B/A}$ is a coh. seq.
 (dist. triangle).

pf: Choose a free resolution
 $P. \rightarrow A.$

Then
 Can be chosen as a free map. (for ex. bar res. of $P./B$)

$$\begin{array}{ccc} P. & \xrightarrow{\sim} & A \\ \downarrow & & \downarrow \\ Q. & \xrightarrow{\sim} & B \end{array}$$
 for some $Q.$

Then we have a SES

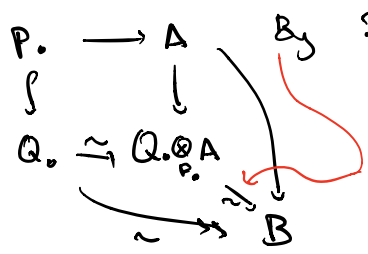
$$\Omega_{P./k} \otimes_{P.} Q. \rightarrow \Omega_{Q./k} \rightarrow \Omega_{Q./P.} \rightarrow 0 \quad (\text{localize})$$

Apply $-\otimes_{Q.} B$:

$$0 \rightarrow \Omega_{P./k} \otimes_{P.} B \rightarrow \Omega_{Q./k} \otimes_{Q.} B \rightarrow \Omega_{Q./P.} \otimes_{Q.} B \rightarrow 0$$

$\cong \quad \cong \quad \cong$
 $L_{A/k} \otimes_A B \quad L_{B/k} \quad L_{B/A}$

Because $P. \rightarrow Q.$ has levelwise retraction



By 2 out of 3: weak equiv.

$$\text{So } L_{B/A} \cong \underbrace{\Omega_{A \otimes_{P.} Q./A} \otimes_{A \otimes_{P.} Q.} B}_{\text{Base change}} \cong \Omega_{Q./P.} \otimes_{Q.} B \quad \square$$

Cor: $k \rightarrow R$, $U \subseteq R$ mult. subset

Then

$$L_{U^{-1}R/k} \cong U^{-1}L_{R/k}$$

Let $r \in k$ and consider

$$k \rightarrow k[x] \xrightarrow{x \mapsto r} k$$

$$\rightsquigarrow k \otimes_{k[x]} L_{k[x]/k} \rightarrow L_{k/k} \rightarrow L_{k/k[x]} \quad \text{coh. seq.}$$

\cong
 $k[x]$

$$\implies L_{k/k[x]} \cong k[x][1].$$

Now consider

$$\begin{array}{ccc} k[x] & \xrightarrow{x \mapsto r} & k \\ \downarrow \text{incl} & \lrcorner & \downarrow \\ k & \rightarrow & k/(r) \end{array}$$

Take a resolution:

$$\begin{array}{ccccc} & & B_0 & \xrightarrow{\sim} & k \\ k[x] & \xleftarrow{\quad} & & \xrightarrow{x \mapsto r} & \\ & & \downarrow & & \downarrow \\ & & A_0 & & k/(r) \\ k & \xleftarrow{\quad} & & \xrightarrow{\quad} & \end{array}$$

$$\begin{aligned}
\text{Then } \Omega_{A_0/k} \otimes_{A_0} k/(r) &\cong \Omega_{B_0/k(r)} \otimes_{B_0} A_0 \otimes_{A_0} k/(r) \\
&= \Omega_{B_0/k(r)} \otimes_{B_0} k \otimes_{k} k/(r) \\
&\cong \Omega_{k/k(r)} \otimes_{k} k/(r) \\
&= \Sigma k/(r) \quad (\Sigma \text{ suspension})
\end{aligned}$$

If $A_0 \rightarrow k/(r)$ was a reg we'd have $\Omega_{k/(r)/k} = \Sigma k/(r)$.

Check this on normalized ch. compl.

Take $NB_0 = \{k(x) \xrightarrow{x} k(x)\}$

Then $NA_0 = \{k \xrightarrow{r} k\}$. This is a k -resolution of $k/(r)$ iff r is not a zero divisor.

Thus $\Omega_{k/(r)/k} \cong \Sigma k/(r)$ if r is not a zero divisor.

Cor: If $k \rightarrow R$ is locally a complete intersection (reg. seq.) then $\text{flat dim}(L_{R/k}) \leq 1$.

Converse true (Noether) by Lichtenbaum-Schlesinger.

IV Vojta's: k Noether. R finite type.

$L_{R/k} \cong 0 \iff R/k$ étale.
f.d. $L_{R/k} = 0 \iff R/k$ smooth

Conjecture: (Quillen) If \dim of $L_{R/k}$ is finite then
that $\dim(L_{R/k}) \leq 2$

proven in a special case by Aramov.