1. Model str. on S
2. Derived functors
3. Simpl. (model) categories.

1.

Call $f: X \longrightarrow Y$ in S a

* <u>weak equiv</u> if $|f|: |X| \longrightarrow |Y|$ is a weak equiv.
* <u>fibration</u> if it is a <u>Kan fibration</u>:

$$\begin{tikzcd} \Lambda^k[n] \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \Delta[n] \arrow[r] \arrow[ur, dashed, "\exists!"] & Y \end{tikzcd}$$

* <u>cofibration</u> if it is an injection.

<u>Remark</u>: A fibrant simplicial set is a <u>Kan complex</u>. It is a model for ∞-groupoids.

__Remark:__ For Kan complexes there is a purely combinatorial description of

$$\pi_i(K, x_0)$$

② Derived functors

__Def:__ Let $F : \mathcal{M} \longrightarrow \mathcal{D}$ ($\mathcal{M}$ model cat.)
A __left derived functor__ of F, $\mathbb{L}F$

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow t"] & \mathcal{D} \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed, "\mathbb{L}F"'] & \end{tikzcd} \quad \text{s.t.} \; \forall \quad \begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow s"] & \mathcal{D} \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, "G"] \arrow[ur, bend right=60, "\mathbb{L}F"'] \arrow[ur, phantom, bend right=30, "\exists u \Rightarrow"] & \end{tikzcd}$$

$\exists!\, u$ s.t.

$$\begin{tikzcd} G\gamma \arrow[rr, Rightarrow, "u\gamma"] \arrow[dr, Rightarrow, "s"'] & \arrow[d, phantom, "\textcircled{\scriptsize =}"] & \mathbb{L}F\gamma \arrow[dl, Rightarrow, "t"] \\ & F & \end{tikzcd}$$

Suppose F __homotopical__: maps weak equiv to iso., then

$$\begin{tikzcd} \mathcal{M} \arrow[rr, "F"] \arrow[dr, "\gamma"'] & \arrow[d, phantom, "\Uparrow \mathrm{id}"] & \mathcal{D} \\ & \mathrm{Ho}(\mathcal{M}) \arrow[ur, dashed, "\mathbb{L}F"'] & \end{tikzcd}$$

Prop 1: Let $\mathcal{M} \xrightarrow{F} \mathcal{D}$ be a functor that maps weak equiv. betw. cofibrant obj. to iso's, then F has a <u>left derived functor</u> and

$$t_X : LFX \longrightarrow FX \quad \text{is}$$

the identity if X is cofibr.

pf: $Q : \mathcal{M} \longrightarrow \mathcal{M}$ cofibr. repl.

write $p : Q \Rightarrow id$

homotopical $\rightsquigarrow$ factors through $Ho(\mathcal{M})$

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "Q"] \arrow[d, "\gamma"'] \arrow[rr, bend left] & \mathcal{M} \arrow[r, "F"] \arrow[d, phantom, "\Uparrow id"] & \mathcal{D} \\
Ho(\mathcal{M}) \arrow[urr, bend right, "LF"'] & &
\end{tikzcd}$$

$t : LF\gamma \Rightarrow F$;

$$t_X : LF\gamma X = FQX \xrightarrow{F(p_X)} FX$$

Claim:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Uparrow t"] & \mathcal{D} \\
Ho(\mathcal{M}) \arrow[ur, "LF"'] &
\end{tikzcd}$$

is LDF

Def: Now let

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "\Nearrow"] & N \arrow[d, "\delta"] \\
Ho(\mathcal{M}) \arrow[r, "\mathbb{L}F"] & Ho(N)
\end{tikzcd}$$

A <u>total left derived functor</u> is a $\mathbb{L}F$ s.t. $\mathbb{L}F$ is a LDF of δF.

Thm (Quillen's TDF thm): Let

$$F: M \rightleftarrows N : G$$ be an adjunction

s.t. either

* F pres. cofib., G pres fib.
* — " — (trivial) cofib.
* G — " ——— fib.
* F pres. weak equiv. on M_c and G ——— " ——— N_f

then

$$\mathbb{L}F : M \rightleftarrows N : \mathbb{R}G$$

exists and form an adjunction and furthermore, if the following holds for all $A \in M_c$, $X \in N_f$:

$$W \ni (FA \to X) \iff (A \to GX) \in W$$

$\mathbb{L}F, \mathbb{R}G$ is an adjoint equiv. of cats.

In this situation we say that (F, G) is a <u>Quillen adjunction</u>

resp. <u>Quillen equivalence</u>.

Ex: Geometric realization $|-| : (sSet) \rightleftarrows (Top) : S$ is a Quillen equivalence.

Sketch: for a space X the counit map

$$|SX| \longrightarrow X \quad \text{is a weak eq.}$$

$$\pi_n(|SX|, x) \simeq \pi_n(SX, x)$$

$$(\Delta^n \to X) \simeq (|\Delta[n]| \to X)$$
$$\simeq (\Delta[n] \longrightarrow SX) \quad SX \text{ Kan comple.}$$

Cochain complexes Let R be a ring

$$M := Ch^*_{\geq 0}(R), \quad (C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} \cdots)$$

Model str:

$$W = (\text{quasi-iso.})$$
$$F = (\text{surjections} - \text{inj. kernel})$$
$$C = (\text{injections at deg} \geq 1)$$

Called: the injective model structure.

Remark: Suppose I fibrant:

then
$$\begin{tikzcd} I^0 \arrow[r] \arrow[d] & I^1 \arrow[r] \arrow[d] & I^2 \arrow[r] \arrow[d] & \cdots \\ * \arrow[r] & * \arrow[r] & * \arrow[r] & \cdots \end{tikzcd}$$

I is injectives.

Let M be an R-mod. and $M[0] \longrightarrow I$ a fibrant repl.

$$\begin{tikzcd} M \arrow[r] \arrow[d, "C \cap \underline{W}"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\ I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

$M = \ker(I^0 \longrightarrow I^1)$ and $M \hookrightarrow I^0$.

$\mathrm{Hom}_R(M,-) : \mathcal{M} \longrightarrow Ch_{\geq 0}(\mathbb{Z})$

By prop. 1 this has a total right derived functor

$$\mathbb{R}\mathrm{Hom}_R(M,-) \quad \text{and}$$

$$H^i \mathbb{R}\mathrm{Hom}_R(M, N[0]) = H^i \mathrm{Hom}(M, I^\bullet)$$
$$= \mathrm{Ext}^i(M,N)$$