

(Jeroen) Model categories (2 lectures)

- Plan:
- ① Mod. cats
 - ② Top
 - ③ Homotopy Cat.
 - ④ sSet
 - ⑤ derived functors
 - ⑥ Cochain complexes
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① Def.: A model category M is a category with three classes of morph:

W : weak equiv.

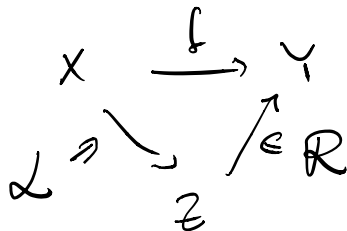
C : cofibrations

F : fibrations

such that

1. M is (finitely) (co)complete
2. W contains all iso's and satisfies the 2-out-of-3 property.
3. $(C, W \cap F)$ and $(C \cap W, F)$ are weak factorization systems. (functorial)

Def: (L, R) is a weak factorization system WFS if \forall



$$L = \{ \text{maps } \dashv \text{ LLP w.r.t } R \}$$

$$R = \{ \text{---} \dashv \text{ RLP ---} L \}$$

Lemma:

Cor:

Def: A an obj. in \mathcal{M}

$(0 \rightarrow A) \in \mathcal{C} : A$ cofibrant

$(A \rightarrow 1) \in \mathcal{F} : A$ fibrant.

(Co)fibrant replacement:

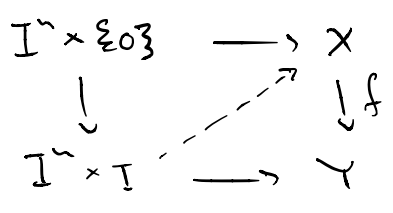


② Top

Ex: \mathcal{W} = weak homot. equiv.

\mathcal{C} = retracts of rel. cell complexes $X \rightarrow Y$

\mathcal{F} = Serre fibrations



③ Homotopy category

Def: • A cylinder object is a diagram

$$\begin{array}{ccc}
 A \amalg A & & \Delta \\
 \downarrow i & \sigma & \searrow \\
 A \wedge I & \xrightarrow{\sigma \in W} & A
 \end{array}$$

• A left homotopy: $(f \sim g): A \rightarrow B$ is

$$\begin{array}{ccc}
 A \amalg A & & (f, g) \\
 \downarrow i & & \searrow \\
 A \wedge I & \xrightarrow{h} & B
 \end{array}$$

• A path object for B is

$$\begin{array}{ccc}
 & \xrightarrow{w \circ s} & B^I \\
 B & & \downarrow p \circ F \\
 & \searrow & B \times B
 \end{array}$$

• A right homotopy $(f \sim g): A \rightarrow B$

$$\begin{array}{ccc}
 & \xrightarrow{s} & B^I \\
 A & & \downarrow h \\
 & \searrow (f, g) & B \times B
 \end{array}$$

Prop: $A = \text{cofibr}$

$B = \text{fibr.}$

$(f \overset{L}{\sim} g) \iff (f \overset{R}{\sim} g)$ and we
write just \sim .

Def: The homotopy category $\text{Ho } M$
of M is the category with the
same obj. as M and $\forall X, Y \in \text{Ob}(M)$

fibrant and cofibrant

$$\text{Ho}(M)(X, Y) := M(\mathbb{R}QX, \mathbb{R}QY) / \sim$$

Thm: There is a functor

$$\gamma: M \longrightarrow \text{Ho } M$$

• which is the id on obj. and
• " " makes $\text{Ho } M$ the loc. of M
at \mathcal{W} :

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\text{iso.}} & \mathcal{D} \\ M & \longrightarrow & \mathcal{D} \\ \gamma \downarrow & & \nearrow \exists! \\ \text{Ho } M & & \end{array}$$

$$\implies (X \xrightarrow{f} Y) \in \mathcal{W} \iff \gamma(f) \text{ is an iso.}$$

(proof in probos)

④ Simplicial sets

Def: Δ is the category ~

$$\text{obj: } [n] = \{0, \dots, n\} \quad (n \geq 0)$$

Special morph: $d^i: [n-1] \rightarrow [n]$ "add" i

$s^j: [n] \rightarrow [n-1]$ "hit j twice"

Def: A simplicial obj. in a cat \mathcal{C} is a functor

$$X: \Delta^{\text{op}} \longrightarrow \mathcal{C}$$

Remark: a simplicial object is compl. determined by its face and degen. maps:

maps:

$$X_0 \begin{array}{c} \xleftarrow{d_0} \\ \xrightarrow{s_0} \\ \xleftarrow{d_1} \end{array} X_1 \begin{array}{c} \xrightarrow{s_1} \\ \xleftarrow{d_1} \\ \xrightarrow{s_2} \\ \xleftarrow{d_2} \end{array} X_2 \dots$$

Def: $\Delta[-]: \Delta \longrightarrow \mathcal{S} := (\text{set})^{\Delta^{\text{op}}}$
 $[n] \longmapsto \Delta(-, [n])$ (Repr. functor)

Remark: $X \in \mathcal{S} \Rightarrow X_n \cong \mathcal{S}(\Delta[n], X)$

$$X \cong \text{colim}_{\Delta[n] \rightarrow X} \Delta[n]$$

Geo. Realization:

$$|X| := \operatorname{colim}_{\Delta[n] \rightarrow X} \Delta^n$$

$$\left(\begin{array}{l} \Delta^n \in \mathbb{R}^{n+1} \\ \text{standard} \\ n\text{-simplex} \end{array} \right)$$

Singular Complex Y space

$$SY: \Delta^p \longrightarrow (\text{set})$$

$$[n] \longmapsto \operatorname{Top}(\Delta^n, Y)$$

Prop: \exists adjunction

$$\operatorname{Top}(|X|, Y) \simeq \mathcal{S}(X, SY)$$