(Jeroen) <u>Model categories</u> (2 lectures)

<u>Plan</u>: 1. Mod. cats
2. Top
3. Homotopy Cat.
4. sSet
5. Derived functors
6. Cochain complexes

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1. <u>Def.</u> A model category M is a category with three classes of morph:
W : weak equiv.
C : cofibrations
F : fibrations
such that
1. M is (finitely) (co) complete
2. W contains all iso's and satisfies the 2-out-of-3 property.
3. $(C, W \cap F)$ and $(C \cap W, F)$ are weak factorization systems. (functorial)

Def: $(\mathcal{L}, \mathcal{R})$ is a weak factorization system WFS if $\forall$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathcal{L} \ni"'] & & Y \\ & Z \arrow[ur, "\in \mathcal{R}"'] & \end{tikzcd}$$

$\mathcal{L} = \{$ maps w/ LLP wrt $\mathcal{R} \}$

$\mathcal{R} = \{$ — " — RLP — " — $\mathcal{L} \}$

Lemma:

Cor:

Def: A an obj. in $\mathcal{M}$

$(0 \to A) \in \mathcal{C}$: A cofibrant
$(A \to 1) \in \mathcal{F}$: A fibrant.

(Co)fibrant replacement:

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr, "\mathcal{C} \ni"'] & & X \\ & QX \arrow[ur, "\in \mathcal{F} \cap \mathcal{W}"'] & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr] \arrow[dr, "\mathcal{C} \cap \mathcal{W}"'] & & 1 \\ & RX \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

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② Top

Ex: $\mathcal{W}$ = weak homot. equiv.
$\mathcal{C}$ = retracts of rel. cell complexes $X \to Y$
$\mathcal{F}$ = Serre fibrations

$$\begin{tikzcd} I^n \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ I^n \times I \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

③ Homotopy category

Def: • A *cylinder object* is a diagram

$$\begin{tikzcd} A \amalg A \arrow[d, "C \ni i"'] \arrow[drr, bend left, "\nabla"] \arrow[dr, phantom, "\circlearrowright"] & & \\ A \wedge I \arrow[rr, bend right, "\sigma \in W"'] & & A \end{tikzcd}$$

• A *left homotopy* $(f \overset{l}{\sim} g) : A \longrightarrow B$ is

$$\begin{tikzcd} A \amalg A \arrow[d, "i"'] \arrow[dr, bend left, "(f,g)"] & \\ A \wedge I \arrow[r, bend right, "h"'] & B \end{tikzcd}$$

• A *path object* for B is

$$\begin{tikzcd} & B^I \arrow[d, "p \in F"] \\ B \arrow[ur, "W \ni s"] \arrow[r, bend right, "\nabla"'] & B \times B \end{tikzcd}$$

• A *right homotopy* $(f \overset{r}{\sim} g) : A \longrightarrow B$

$$\begin{tikzcd} & B^I \arrow[d, "h"] \\ A \arrow[ur, "s"] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd}$$

Prop: A: cofibrant
B: fibr.
$(f \overset{l}{\sim} g) \longleftrightarrow (f \overset{r}{\sim} g)$ and we write just $\sim$.

Def: The homotopy category $\operatorname{Ho} M$ of M is the category with the same obj. as M and $\forall X, Y \in Ob(M)$

fibrant and cofibrant

$$\operatorname{Ho}(M)(X, Y) := M(RQX, RQY)/\sim$$

Thm: There is a functor

$$\gamma : M \longrightarrow \operatorname{Ho} M$$

- which is the id on obj. and
- –"– makes $\operatorname{Ho} M$ the loc. of M at W:

$$\begin{tikzcd} M \arrow[r, "W \mapsto \text{iso.}"] \arrow[d, "\gamma"'] & D \\ \operatorname{Ho} M \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$\Rightarrow \left((X \xrightarrow{f} Y) \in W \iff \gamma(f) \text{ is an iso.} \right)$

(proof in photo)

4 Simplicial sets

Def: Δ is the category w

obj: $[n] = \{0, \dots, n\}$ $\quad (n \geq 0)$

Special morph: $d^i : [n-1] \to [n]$ "add" i

$s^j : [n] \to [n-1]$ "hit j twice"

Def: A simpl. obj. in a cat $\mathcal{C}$ is a functor

$$X : \Delta^{op} \longrightarrow \mathcal{C}$$

Rmk: a simplicial object is compl. determined by its face and degen. maps:

$$X_0 \underset{d_1}{\overset{d_0}{\leftleftarrows}} X_1 \quad X_2 \quad \dots$$

Def: $\Delta[-] : \Delta \longrightarrow \mathcal{S} := (\mathrm{set})^{\Delta^{op}}$

$[n] \longmapsto \Delta(-, [n])$ $\quad$ (Repr. functor)

Rmk: $X \in \mathcal{S} \implies X_n \simeq \mathcal{S}(\Delta[n], X)$

$$X \cong \operatorname*{colim}_{\Delta[n] \to X} \Delta[n]$$

Geo. Realization:

$$|X| := \operatorname{colim}_{\Delta[n] \to X} \Delta^n$$

($\Delta^n \subseteq \mathbb{R}^{n+1}$ standard n-simplex)

Singular Complex $\quad Y$ space

$$SY : \Delta^{op} \longrightarrow (\text{set})$$
$$[n] \longmapsto \mathrm{Top}(\Delta^n, Y)$$

Prop: $\exists$ adjunction
$$\mathrm{Top}(|X|, Y) \simeq \mathcal{S}(X, SY)$$