I. Persp.

Def. Ω = linearization

$$\begin{array}{ccc} X & \longmapsto & \Omega_X \\ & & \mathcal{O}_X\text{-mod.} \\ \text{scheme} & \longmapsto & \mathbb{L}_X \in D^{(-\infty,0]}(X) \end{array}$$

tangent spaces, derivatives

II. Deformation situation

$$I \subset A' \twoheadrightarrow A = A'/I \quad \text{rings} \qquad I^2 = 0$$

$I^2 = 0 \implies I$ and A-module

$$(\quad \bar{a}.x = (a+x').x = ax + x'x = ax \quad)$$

A' def. of A

Rmk: $A' \twoheadrightarrow A \twoheadrightarrow A_0$, $A = A'/I$, $A_0 = A'/J$, $IJ = 0 \implies$

I is an A_0-module.

Ex: $I \subset A \twoheadrightarrow A/I^{N-1} \twoheadrightarrow A/I^{N-2} \twoheadrightarrow \cdots \twoheadrightarrow A/I = A_0$

$I^N = 0$ $\qquad I^n/I^{n+1}$ $\qquad A_0$-modules

Ex: Trivial def. A ring, M A-mod.

$$\begin{array}{c} A[M] = A \oplus M \\ \downarrow \\ A \end{array} \qquad M^2 = 0$$

trivial deformation since

$$A \hookrightarrow A[M]$$

Ex: $k[\varepsilon]/(\varepsilon^3) \longrightarrow k[\varepsilon]/(\varepsilon^2)$ non-trivial def. (no section)

Schemes

Inf. def. of schemes: Closed imm.

$$j : X \hookrightarrow X' \quad \text{given by}$$

$$I \subset \mathcal{O}_{X'} \quad \text{s.t.} \quad I^2 = 0$$

$$\left(\Rightarrow \quad I \xrightarrow{\sim} j_* j^* I \; : \; \text{identify } I \text{ w/ } j^* I \in QCoh(X) \right)$$

Ex: Triv. def.

$$\begin{tikzcd} X \arrow[r, hook, "M", "i"'] \arrow[rr, bend right, "id"'] & X[M] = \operatorname{Spec} \mathcal{O}_X[M] \arrow[r, "r"] & X \end{tikzcd}$$

Tangent / Cotangent calculation

$$(*)\quad \begin{tikzcd} \operatorname{Spec} k \arrow[r, "x"] \arrow[d] & X \\ \operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \arrow[ur, dashed] & \end{tikzcd}$$

$$\begin{tikzcd} k & \mathcal{O}_x \arrow[l] \arrow[dl, dashed] & & \\ k[\varepsilon]/(\varepsilon^2) \arrow[u] & & \mathfrak{m}_x \arrow[dl] & \\ & (\varepsilon) & & \mathfrak{m}_x^2 \arrow[dl] \\ & & (\varepsilon^2) = (0) & \end{tikzcd}$$

$\mathcal{O}_x \supset \mathfrak{m}_x \supset \mathfrak{m}_x^2$

Two questions:

1. Given x and $I \in QCoh(X)$, find all $X \overset{I}{\hookrightarrow} X'$.
2. Given

$$\begin{tikzcd} X \arrow[r, hook, "I"] \arrow[d] & X' \arrow[dl, dashed] \\ Y & \end{tikzcd}$$

find all lifts

translates to:

$$\begin{tikzcd} 0 & \mathcal{O}_x/\mathfrak{m}_x \arrow[l] \arrow[d] & \mathcal{O}_x/\mathfrak{m}_x^2 \arrow[l, two heads] \arrow[d, dashed] & \mathfrak{m}/\mathfrak{m}^2 \arrow[l] \arrow[d, dashed] & 0 \arrow[l] \\ 0 & k \arrow[l] & k[\varepsilon]/(\varepsilon^2) \arrow[l, two heads] & (\varepsilon) \arrow[l] & 0 \arrow[l] \end{tikzcd}$$

If X is a k-scheme

$$\mathcal{O}_x/\mathfrak{m}^2 = k[\mathfrak{m}/\mathfrak{m}^2]$$

so

$$(*) \iff \operatorname{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, (\varepsilon)) = (\mathfrak{m}/\mathfrak{m}^2)^\vee$$

Zariski tang. sp.

<u>Aside:</u> Cotangent complex for schemes

$$\begin{array}{c} X \\ \downarrow f \\ S \end{array} \quad \text{scheme}$$

$SCR(f^{-1}\mathcal{O}_S) \ni (f^{-1}\mathcal{O}_S \longrightarrow \mathcal{O}_X)$
resolve $\mathcal{O}_X$ w/ free/colim. $f^{-1}\mathcal{O}_S$-algebras + take $\Omega_{-/f^{-1}\mathcal{O}_S}$.

$$\rightsquigarrow \quad \mathbb{L}_{X/S} \in D^{(-\infty,0]}_{QCoh}(X) \underset{\uparrow \text{often}}{=} D^{(-\infty,0]}(QCoh\, X)$$

$$H^0(\mathbb{L}_{X/S}) = \Omega_{X/S}.$$

$$X \xrightarrow{f} Y \longrightarrow Z \qquad \rightsquigarrow$$

$$f^*\mathbb{L}_{Y/Z} \longrightarrow \mathbb{L}_{X/Z} \longrightarrow \mathbb{L}_{X/Y} \quad \text{exact triangle.}$$

<u>III tangent spaces via $\mathbb{L}$</u>

$$\begin{tikzcd} \operatorname{Spec} k' \arrow[r, "x"] & X \arrow[d] \\ & \operatorname{Spec} k \end{tikzcd} \qquad k' = k(x)$$

$$(+) \quad x^*\mathbb{L}_{X/k} \longrightarrow \mathbb{L}_{k'/k} \longrightarrow \mathbb{L}_{\operatorname{Spec} k'/X}$$

Lemma: $Z \hookrightarrow X$ closed imm. given by ideal I. Then $\Omega_{Z/X} = 0$

$$H^{-1}(\mathbb{L}_{Z/X}) = I/I^2 = j^*I,$$

the <u>conormal sheaf</u> $N^\vee_{Z/X}$

Prop: (X Noetherian)

$$\mathbb{L}_{Z/X} \simeq N^\vee_{Z/X}[1]$$

iff $Z \hookrightarrow X$ is a regular embedding (locally $I = (f_1, \ldots, f_n)$ regular sequence)

LES of (+):

$$\begin{array}{llllll} H^{-1} & & & 0 \longrightarrow & m/m^2 & \longrightarrow \\ H^0 & \longrightarrow \Omega_{X/k}|_x & \longrightarrow 0 & \longrightarrow 0 & & \text{so} \end{array}$$

$$\Omega_{X/k} \otimes k(x) \cong m/m^2$$

Aside: k'/k field ext. $\mathbb{L}_{k'/k} = 0$ if k'/k alg + sep.

IV Formal smoothness (/étale/unram.)

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook, "I"'] & X \arrow[d] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed, "g'"] & Y \end{tikzcd}$$

$$\text{"} A'/I \text{"}$$

In general $I^N = 0$

$$\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'/I^2 \hookrightarrow \cdots \hookrightarrow \operatorname{Spec} A'$$

all sq. zero.

Enough to consider $I^2 = 0$.

Def: $X \to Y$ is

- formally smooth if $\exists g'$ $\forall A, A', I, g$
- — " — étale — " — $\exists! g'$ — " —
- — " — unram. — " — $\exists$ at most one such g' — " —.

Def: $X \to Y$ is

- smooth if form. sm. + loc. fin. pres
- étale if — " — ét. — " —
- unram. if — " — unram + loc. fin. type

Fact. f smooth $\Longleftrightarrow$ loc. fin. pres. + $\mathbb{L}_{X/Y} \simeq \Omega_{X/Y}[0]$ loc. free of fin. rank
$\Longleftrightarrow$ f flat fin. pres. of rel. dim. r and $\Omega_{X/Y}$ loc. free of rank r.

Problem:

$$\left\{ \begin{tikzcd} Z \arrow[r] \arrow[d, "I"'] & X \\ Z' \arrow[ur, dashed] & \end{tikzcd} \right\} \Big/ \sim \;=:\; \mathrm{Exal}_X(Z, I)$$

$\parallel$ Thm ?

$$\mathrm{Ext}^1(\mathbb{L}_{Z/X}, I)$$

$$\left\{ \begin{tikzcd} Z \arrow[r] \arrow[d, "I"'] & X \\ Z' \arrow[ur] & \end{tikzcd} \right\} =: \mathrm{Der}_X(Z, I)$$

$$= \mathrm{Ext}^0(\mathbb{L}_{Z/X}, I)$$

Problem 2:

$$\alpha = \left(\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "I"'] & X \arrow[d] \\ Z' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \right)$$

Obstruction:

$$o(\alpha) \in \mathrm{Ext}^1_{\mathcal{O}_Z}(Lf^* \mathbb{L}_{X/Y}, I)$$

lift $\exists \iff o(\alpha) = 0$

If $o(\alpha) = 0$, then the set of lifts is (non-can.) bij. to

$$\mathrm{Ext}^0_{\mathcal{O}_Z}(Lf^*\mathbb{L}_{X/Y}, I)$$

$$\begin{tikzcd}
\operatorname{Spec} A \arrow[r] \arrow[d, hook, "I"'] & X \arrow[d, "\text{smooth}"] \\
\operatorname{Spec} A' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd} \iff o(\alpha) \in \mathrm{Ext}^1_A(\underbrace{Lf^*\underbrace{\mathbb{L}_{X/Y}}_{\Omega_{X/Y}[0] \text{ loc.fr.}}}_{(f^*\Omega_{X/Y})[0]}, I)$$

$$= \mathrm{Ext}(\mathcal{O}_Z, (f^*\Omega_{X/Y})^\vee \otimes I)$$
$$= H^1(Z, f^*T_{X/Y} \otimes I)$$
$$= 0 \quad \text{since } Z \text{ affine.}$$

Set of lifts:

$$\begin{aligned}
\mathrm{Ext}^0(f^*\Omega_{X/Y}, I) &= \mathrm{Hom}(f^*\Omega_{X/Y}, I) \\
&= \mathrm{Hom}(\Omega_{X/Y}, f_*I) \\
&= \mathrm{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, f_*I)
\end{aligned}$$

__Problem 3:__

$$\alpha = \left(\begin{tikzcd} X \arrow[r, dashed] \arrow[d, "\text{flat}"'] \arrow[dr, phantom, "\square"] & X' \arrow[d, dashed] \\ S \arrow[r, hook, "I"'] & S' \end{tikzcd} \right)$$

$o(\alpha) \in \mathrm{Ext}^2(\mathbb{L}_{X/S}, f^*I) = \mathrm{Obs}(X/S, I)$

Set of def. is an $\mathrm{Ext}^1(\_)$ torsor $\mathrm{Def}(X/S, I)$

any lift has inf aut $= \mathrm{Ext}^0(\_) = \mathrm{Aut}(X/S, I)$

__Ex:__

$$\begin{tikzcd} A[x_1, \ldots, x_n] & A'[x_1, \ldots, x_n] \arrow[l] \\ A'/I = A \arrow[u] & A' \arrow[l] \arrow[u] \end{tikzcd}$$

Aut: $\varphi: \; x_i \longmapsto x_i + \delta_i$, $\delta_i \in IA[x_1, \ldots, x_n]$

$\Omega_{X/S} = A[x_1, \ldots, x_n]\langle e_1, \ldots, e_n \rangle$

$\varphi \longleftrightarrow \sum \delta_i e_i$