

I. Persp.

Def. $\Omega_X =$ linearization

$$\begin{array}{ccc} X & \longrightarrow & \Omega_X \\ \text{scheme} & \searrow & \text{\scriptsize } \mathcal{O}_X\text{-mod.} \\ & & \mathbb{V}_X \in \mathcal{D}^{(-\infty, 0]}(X) \end{array}$$

tangent spaces, derivations

II. Deformation situation

$$I \subset A' \longrightarrow A = A'/I \quad \text{wugs} \quad I^2 = 0$$

$$\begin{array}{l} I^2 = 0 \quad \Rightarrow \quad I \text{ and } A\text{-module} \\ (\bar{a} \cdot x = (a+x') \cdot x = ax + x'x = ax) \\ \quad \quad \quad A' \text{ det. of } A \end{array}$$

Remark:

$$\begin{array}{ccccccc} A' & \longrightarrow & A & \longrightarrow & A_0 & & \\ & & \parallel & & \parallel & & \\ & & A'/I & & A'/J & & \\ & & & & & & IJ=0 \Rightarrow \\ I & \text{is an } & A_0\text{-module.} & & & & \end{array}$$

Ex:

$$\begin{array}{ccccccc} I \subset A & \longrightarrow & A/I^{N-1} & \longrightarrow & A/I^{N-2} & \longrightarrow & \dots \longrightarrow A/I = A_0 \\ I^N = 0 & & I^N/I^{N-1} & & A_0\text{-modules} & & \end{array}$$

Ex: Trivial def. A ring, M A -mod.

$$A[M] = A \oplus M \quad M^2 = 0$$

\downarrow
 A

trivial deformation since
 $A \hookrightarrow A[M]$

Ex: $k[\epsilon]/(\epsilon^3) \rightarrow k[\epsilon]/(\epsilon^2)$ non-trivial def.
(no section)

Schemes

Int. def. of schemes: closed imm.

$j: X \hookrightarrow X'$ given by

$$I \subset \mathcal{O}_{X'} \quad \text{st.} \quad I^2 = 0$$

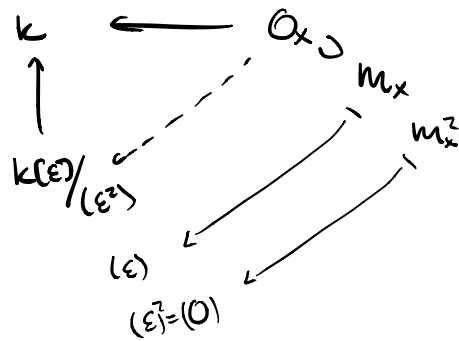
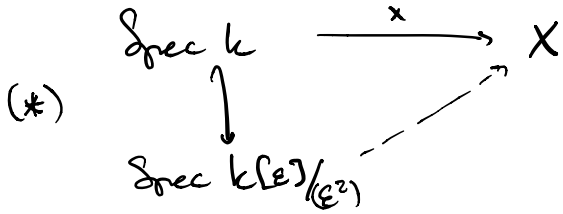
($\Rightarrow I \xrightarrow{\sim} j_* j^* I$: identity $I \xrightarrow{\sim} j_* j^* I$) $\mathcal{O}_{\text{coh}}(X)$

Ex: Div. def.

$$X \xrightarrow[i]{M} X[M] = \text{Spec } \mathcal{O}_X[M] \xrightarrow{r} X$$

id \oplus

Tangent / Cotangent calculation



translates to:

$$\begin{array}{ccccccc} 0 & \leftarrow & \mathcal{O}_x / \mathfrak{m}_x & \leftarrow & \mathcal{O}_x / \mathfrak{m}_x^2 & \leftarrow & \mathfrak{m}_x / \mathfrak{m}_x^2 \leftarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \leftarrow & k & \leftarrow & k[\epsilon]/(\epsilon^2) & \leftarrow & (\epsilon) \leftarrow 0 \end{array}$$

$\iff X$ is a k -scheme

$$\mathcal{O}_x / \mathfrak{m}_x^2 = k[\mathfrak{m}_x / \mathfrak{m}_x^2]$$

so

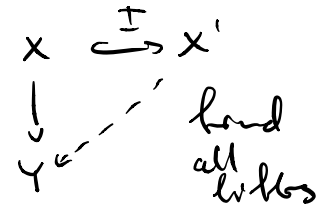
$$(*) \iff \text{Hom}_k(\mathfrak{m}_x / \mathfrak{m}_x^2, (\epsilon)) = (\mathfrak{m}_x / \mathfrak{m}_x^2)^\vee$$

Zariski tang sp.

Two questions:

1. Given x and $I \in \mathcal{O}_{\text{Coh}}(X)$, find all $x \xrightarrow{I} x'$.

2. Given



Aside: Coboundary complex for schemes

X scheme $\text{SCR}(f^{-1}\mathcal{O}_S) \ni (f^{-1}\mathcal{O}_S \rightarrow \mathcal{O}_X)$
 $\downarrow f$ resolve \mathcal{O}_X w/ free/cohor.
 S $f^{-1}\mathcal{O}_S$ -algebras + take
 $\Omega_{-/f^{-1}\mathcal{O}_S}$.

$$\rightsquigarrow \mathbb{L}_{X/S} \in \mathcal{D}_{\text{qcoh}}^{(-\infty, 0]}(X) = \mathcal{D}^{(-\infty, 0]}(\text{Qcoh } X)$$

↑
often

$$H^0(\mathbb{L}_{X/S}) = \Omega_{X/S}.$$

$$X \xrightarrow{f} Y \rightarrow Z \rightsquigarrow$$

$$f^* \mathbb{L}_{Y/Z} \rightarrow \mathbb{L}_{X/Z} \rightarrow \mathbb{L}_{X/Y} \quad \text{exact triangle.}$$

III tangent spaces via \mathbb{L}

$$\begin{array}{ccc}
 \text{Spec } k' & \xrightarrow{x} & X \\
 \text{" } k(x) & & \downarrow \\
 & & \text{Spec } k
 \end{array}$$

$$(+)\quad x^* \mathbb{L}_{X/k} \rightarrow \mathbb{L}_{k'/k} \rightarrow \mathbb{L}_{\text{Spec } k'/X}$$

Lemma: $Z \rightarrow X$ closed imm. given
 by ideal I . Then $\Omega_{Z/X} = 0$
 $H^1(\mathcal{L}_{Z/X}) = I/I^2 = j^*I$,
 We Conormal sheaf $N_{Z/X}^\vee$

Prop: (X Noetherian)

$$\mathcal{L}_{Z/X} \cong N_{Z/X}^\vee(1)$$

iff $Z \rightarrow X$ is a regular embedding
 (locally $I = (f_1, \dots, f_n)$ regular sequence)

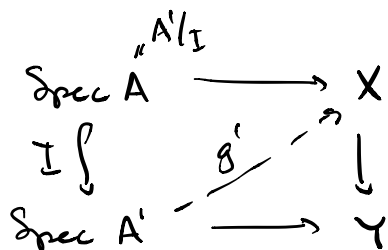
LFS of $(+)$:

$$\begin{array}{ccccccc} H^{-1} & & & 0 & \rightarrow & \mathfrak{m}/\mathfrak{m}^2 & \rightarrow \\ H^0 & \rightarrow & \Omega_{X/k}|_X & \rightarrow & 0 & \rightarrow & 0 \quad \text{so} \end{array}$$

$$\Omega_{X/k} \otimes k(x) \cong \mathfrak{m}/\mathfrak{m}^2$$

Aside: $k|k$ field ext. $\Omega_{k/k} = 0$ if $k|k$
 alg + sep.

IV Formal smoothness (étale/univ.)



In general $I^2 = 0$

$$\text{Spec } A \hookrightarrow \text{Spec } A'/I^2 \hookrightarrow \text{Spec } A'$$

all sq zero.

Enough to consider $I^2 = 0$.

Def: $X \rightarrow Y$ is

- locally smooth if $\exists g' \forall A, A', I, g$
- étale if $\exists! g'$
- univ. if \exists at most one such g' .

Def: $X \rightarrow Y$ is

- smooth if loc. sm. + loc. fm pres
- étale if ét.
- univ. if univ. + loc. fm type

Fact. f smooth \Leftrightarrow loc. free pres. +
 $\mathbb{L}_{X/Y} \simeq \Omega_{X/Y}[0]$ loc. free of ^{high rank}
 $\Leftrightarrow f$ flat free pres. of
 rel. dim. r and
 $\Omega_{X/Y}$ loc. free of rank r .

Problem:

$$\left\{ \begin{array}{ccc} Z & \longrightarrow & X \\ I \downarrow & \nearrow & \\ Z' & & \end{array} \right\} \underset{\sim}{=} \begin{array}{c} \text{Exal}_X(Z, I) \\ \parallel \text{Thm ?} \\ \text{Ext}'(\mathbb{L}_{Z/X}, I) \end{array}$$

$$\left\{ \begin{array}{ccc} Z & \longrightarrow & X \\ I \downarrow & \nearrow & \\ G \downarrow & & \\ Z' & & \end{array} \right\} = \begin{array}{c} \text{Der}_X(Z, I) \\ \\ = \text{Ext}^0(\mathbb{L}_{Z/X}, I) \end{array}$$

Problem 2:

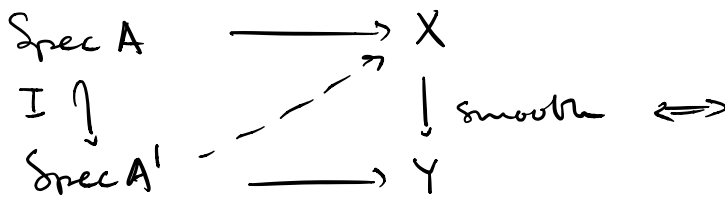
$$\alpha = \left(\begin{array}{ccc} Z & \xrightarrow{f} & X \\ I \downarrow & \nearrow & \downarrow \\ Z' & \longrightarrow & Y \end{array} \right)$$

Obstruction:

$$o(\alpha) \in \text{Ext}_{\mathcal{O}_Z}^1(Lf^* \mathbb{L}_{X/Y}, I)$$

$$\text{liftings} \exists \iff \alpha = 0$$

If $\alpha = 0$, then the set of liftings is (non-can.) bijectively given by $\text{Ext}_{\mathcal{O}_Z}^0(f^*\Omega_{X/Y}, I)$



$$\alpha \in \text{Ext}_A^1(\underbrace{f^*\Omega_{X/Y}}_{\Omega_{X/Y}[0] \text{ local}}, I)$$

$$\underbrace{\hspace{10em}}_{(f^*\Omega_{X/Y})[0]}$$

$$\begin{aligned}
 &= \text{Ext}(\mathcal{O}_Z, (f^*\Omega_{X/Y}) \otimes I) \\
 &= H^1(Z, f^*T_{X/Y} \otimes I) \\
 &= 0 \text{ since } Z \text{ affine.}
 \end{aligned}$$

Set of liftings:

$$\begin{aligned}
 \text{Ext}^0(f^*\Omega_{X/Y}, I) &= \text{Hom}(f^*\Omega_{X/Y}, I) \\
 &= \text{Hom}(\Omega_{X/Y}, f_*I) \\
 &= \text{Der}_{\mathcal{O}_Y}(\mathcal{O}_X, f_*I)
 \end{aligned}$$

Problem 3:

$$\alpha = \left(\begin{array}{ccc} X & \longrightarrow & X' \\ \text{flat} \downarrow & \square & \downarrow \\ S & \xleftarrow{I} & S' \end{array} \right)$$

$o(\alpha) \in \text{Ext}^2(\mathbb{L}_{X/S}, f^* I) = \text{Obs}(X/S, I) = \text{Def}(X/S, I)$

Sub of def. is an $\text{Ext}^1(-)$ torsor

any lobb has int aut = $\text{Ext}^0(-) = \text{Aut}(X/S, I)$

Ex:

$$\begin{array}{ccc} A[x_1, \dots, x_n] & \longleftarrow & A'[x_1, \dots, x_n] \\ \uparrow & & \uparrow \\ k/I = A & \longleftarrow & A' \end{array}$$

Aut: $\mathcal{L}: x_i \longmapsto x_i + \delta_i$
 $\delta_i \in I A[x_1, \dots, x_n]$

$\Omega_{X/S} = A[x_1, \dots, x_n] \langle e_1, \dots, e_n \rangle$

$\mathcal{L} \longmapsto \sum \delta_j e_j$