Homework problems for lecture #6

To be handed in on paper on Mar 17 (lecture #7) or earlier by email.

Throughout X denotes a complex manifold.

(1) Let L be a line bundle on X.
  (a) Show that L is trivial (i.e., isomorphic to $X \times \mathbb{C}^1$) if there exists a nowhere vanishing section.
  (b) Suppose that X is compact and connected. Show that L is trivial if L and $L^\vee$ admit non-trivial sections (i.e., sections not identically zero). Conclude that $\Gamma(\mathbb{P}^n, \mathcal{O}(k)) = 0$ if $k < 0$.

(2) Let $Y \subset X$ be a closed submanifold of codimension at least 2.
  (a) Show that $f \mapsto f|_{X \setminus Y}$ gives a bijection
$$\mathcal{O}_X(X) \to \mathcal{O}_X(X \setminus Y).$$
(You may use Huybrechts Exc 1.1.20 without proof. It can be proven using Hartog's theorem 2.1.6 along the lines of the proof of Proposition 1.1.7.)
  (b) Let L be a line bundle on X. Show that the restriction map
$$L(X) \to L(X \setminus Y)$$
is bijective.
  (c) Let L_1 and L_2 be line bundles on X. Show that a map
$$f \colon L_1|_{X \setminus Y} \to L_2|_{X \setminus Y}$$
extends uniquely to a map $g \colon L_1 \to L_2$. Conclude that if L_1 and L_2 are isomorphic over $X \setminus Y$ then they are isomorphic.

Slogan: *line bundles and sections of line bundles only see features in codimension ≤ 1.*

Hint for (1b)*:* Construct a non-trivial section for the trivial line bundle $L \otimes L^\vee$.

Hint for (2c)*:* This can be proven directly but it could be useful to consider $L_1^\vee \otimes L_2$.