Homework problems for lecture #10

To be handed in by email on Feb 12, 2021.

(0) (Warm-up: you don't need to hand this in.) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be continuous maps between locally compact Hausdorff spaces (e.g., complex manifolds).
   (a) Show that f is proper (i.e., closed with compact fibers) if and only if $f^{-1}(C)$ is compact for every compact $C \subseteq Y$.
   (b) Show that if $g \circ f$ is proper, then so is f.

(1) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be meromorphic maps between complex manifolds. As multivalued functions, there is a composition $g \circ f$.
   (a) Show that $g \circ f$ is a meromorphic map if $Y \subseteq Z$ is an open submanifold and g is the corresponding inclusion.
   (b) Show that $g \circ f$ is a meromorphic map for any g such that $g \circ f$ is singlevalued on a dense open subset of X. You probably have to use:

   Theorem (Remmert's proper mapping theorem)**.** *Let $h\colon X \longrightarrow Y$ be a morphism between complex manifolds. If $Z \subseteq X$ is an analytic set such that $h|_Z$ is proper, then $h(Z) \subseteq Y$ is an analytic set.*

   Lemma. *If Z_1 and Z_2 are analytic subsets of a complex manifold X, then $\overline{Z_1 \smallsetminus Z_2}$ is an analytic subset of X.*

(2) Let X be a complex manifold. Show that there is a natural bijection between meromorphic functions on X (Lecture 2) and meromorphic maps $X \dashrightarrow \mathbb{P}^1$ that are not constant with value $\infty = (1:0)$.

(3) Let X be a connected complex manifold, let L be a line bundle on X, and let $V \subseteq \Gamma(X, L)$ be a linear system (of dimension > 0). This defines a holomorphic map $X \smallsetminus \mathrm{Bs}(V) \longrightarrow \mathbb{P}(V)$. Show that this extends uniquely to a meromorphic map $X \dashrightarrow \mathbb{P}(V)$ (with indeterminacy locus contained in the base locus).

Hint for (1a)*:* Use (0b).
Hint for (1b)*:* First consider a suitable analytic set of $X \times Y \times Z$, proper over X by (0b). Its image in $X \times Z$ is perhaps not quite equal to $\Gamma_{g \circ f}$ but close enough.