COMPACTIFICATION OF STACKS AND EXTENDING STACKINESS ACROSS THE BOUNDARY

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Abstract. We give an overview of three different compactification results for Deligne–Mumford stacks. We also give an example demonstrating that compactifications with a prescribed coarse moduli space need not exist.

Introduction

We give an exposition of three compactification results for Deligne–Mumford stacks that generalize Nagata's compactification theorem for schemes. In characteristic zero, they respectively compactify

(i) quasi-projective Deligne–Mumford stacks (over a field);
(ii) separated Deligne–Mumford stacks of finite type that are normal and global quotients; and
(iii) separated Deligne–Mumford stacks of finite type.

The first result does not carry over to positive characteristic (partly due to the usage of resolution of singularities). In arbitrary characteristic, the second result holds if we allow stacks that are not quite Deligne–Mumford. The third result holds in arbitrary characteristic when restricted to tame Deligne–Mumford stack.

We also give an example of a smooth quasi-projective Deligne–Mumford stack $\mathscr{X}$ of dimension three with smooth coarse moduli space X and a normal compactification $\overline{X}$ such that there does not exist a Deligne–Mumford compactification $\overline{\mathscr{X}}$ with coarse space $\overline{X}$. It is thus not always possible to "extend the stack structure over the boundary".

In this paper, all algebraic stacks, algebraic spaces and schemes are assumed to be quasi-compact with quasi-compact and separated diagonal.

Quasi-projective stacks. Recall that a stack $\mathscr{X}$ is a *global quotient stack* if there exists an algebraic space U with an action of GL_n such that $\mathscr{X} = [U/\mathrm{GL}_n]$ [EHKV01]. Furthermore, recall that if k is a field of characteristic zero, then a Deligne–Mumford stack $\mathscr{X}/k$ is *quasi-projective* (resp. *projective*) [Kre09] if

(i) $\mathscr{X}/k$ is separated and of finite type,
(ii) the coarse moduli space X/k is quasi-projective (resp. projective), and
(iii) $\mathscr{X}$ is a global quotient stack.

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A. Kresch observed that quasi-projective stacks are easily compactified using F. Kirwan's resolution of the strictly semi-stable locus [Kir85].

Theorem A (Kresch [Kre09, Thm. 5.3])**.** *Let $\mathscr{X}$ be a Deligne–Mumford stack over a field k of characteristic zero. The following are equivalent:*

(i) *$\mathscr{X}$ is quasi-projective.*
(ii) *There is a projective stack $\overline{\mathscr{X}}/k$ and a dense open immersion $\mathscr{X} \subset \overline{\mathscr{X}}$.*
(iii) *There is a smooth projective Deligne–Mumford stack $\mathscr{P}$ and an immersion $\mathscr{X} \hookrightarrow \mathscr{P}$.*

Moreover, if $\mathscr{X}$ is smooth with quasi-projective coarse space, then $\mathscr{X}$ is quasi-projective [EHKV01, KV04, Jon03], cf. Remark (1.1).

Normal global quotient stacks. The second compactification result is due to D. Edidin (unpublished). We say that an algebraic stack $\mathscr{X}$ is an E-stack if there exists an algebraic space Z and a finite surjective morphism $\pi\colon Z \to \mathscr{X}$ which is étale over a dense open substack. Every Deligne–Mumford stack is an E-stack [Ryd14, Thm. B] and an E-stack has quasi-finite and separated diagonal. Thus, in characteristic zero, the E-stacks are exactly the Deligne–Mumford stacks.

Theorem B (Edidin 2009)**.** *Let S be an excellent algebraic space and let $\mathscr{X} \to S$ be an algebraic stack. The following are equivalent.*

(i) *$\mathscr{X}$ is a normal global quotient E-stack and $\mathscr{X} \to S$ is separated and of finite type.*
(ii) *There exists a normal global quotient E-stack $\overline{\mathscr{X}}$ that is proper over S together with a dense open immersion $\mathscr{X} \subset \overline{\mathscr{X}}$.*

Note that we do not have much control over the boundary. For example, in positive characteristics, it may happen that $\overline{\mathscr{X}}$ is not Deligne–Mumford even if $\mathscr{X}$ is Deligne–Mumford.

Tame Deligne–Mumford stacks. The third compactification result does not require normality or global quotient presentations but can only handle *tame* Deligne–Mumford stacks. Tameness is of course automatic in characteristic zero. In positive characteristic, there are some evidence of the existence of wild Deligne–Mumford stacks that cannot be compactified by Deligne–Mumford stacks.

Theorem C ([Ryd09])**.** *Let $\mathscr{Y}$ be a Deligne–Mumford stack and let $f\colon \mathscr{X} \to \mathscr{Y}$ be a morphism of Deligne–Mumford stacks. The following are equivalent:*

(i) *$\mathscr{X} \to \mathscr{Y}$ is separated, of finite type and strictly tame.*
(ii) *There exists a Deligne–Mumford stack $\overline{\mathscr{X}} \to \mathscr{Y}$ that is proper and tame and an open dense immersion $\mathscr{X} \subseteq \overline{\mathscr{X}}$.*

A morphism of Deligne–Mumford stacks $\mathscr{X} \to \mathscr{Y}$ is *tame* if the relative stabilizer group $\operatorname{Aut}_{\mathscr{X}/\mathscr{Y}}(x)$ has order prime to the characteristic of the residue field for every point $\operatorname{Spec} k \to \mathscr{X}$. Strict tameness, is a refined notion taking care of specializations in mixed characteristics, cf. [Ryd09, Def. 1.2] for details.

Extending stack structure. We give examples of a proper scheme X, an open subscheme $U \subseteq X$ and a Deligne–Mumford stack $\mathscr{U}$ with coarse moduli space U, such that there does not exist a compactification $\mathscr{U} \subseteq \mathscr{X}$ with coarse space X.

Theorem D (2009)**.** *There exists a normal affine variety X of dimension 3, an open smooth subscheme $U \subseteq X$ and a smooth Deligne–Mumford stack $\mathscr{U}$ with coarse space U such that there does not exists a Deligne–Mumford stack $\mathscr{X}$ with coarse space X that contains $\mathscr{U}$ as an open substack. Furthermore, it is possible to find such a $\mathscr{U}$ which is either an orbifold or a gerbe over U.*

In the examples we give, X is the non-simplicial toric variety $xy = zw$ in $\mathbb{A}^4$ and U is the complement of the singularity at the origin. Moreover, in our examples, there does exist an *Artin stack* $\mathscr{X}$ with *good moduli space* X. There is also a Deligne–Mumford compactification of $\mathscr{U}$ with coarse moduli space X' where $X' \to X$ is a resolution of the singularity.

Likewise, even if X is a smooth scheme and $\mathscr{U}$ is a smooth stack with coarse space U which is open in X, it seems unlikely that there always is a Deligne–Mumford compactification $\mathscr{X}$ with coarse space X.

In recent years, there has been an increased interest in variants of the minimal model program for moduli spaces, see for example [Has03, HH09, HH13, CC11, Smy13] and the references therein. In particular, one looks for models with modular interpretations. Since many moduli spaces are Deligne–Mumford stacks, it is natural to run the minimal model program on the level of stacks. Theorem D indicates that a naive generalization of the contraction theorem fails in general: an extremal ray that is contractible on the level of coarse spaces need not be contractible on the level of Deligne–Mumford stacks.

A solution to this conundrum would be to use *Artin stacks*—with infinite stabilizer groups—and good moduli spaces instead of coarse moduli spaces. This approach is closely related to variation of GIT. In the first step of the minimal model program for $\overline{\mathscr{M}_g}$, the model is a Deligne–Mumford stack: the moduli stack of pseudostable curves [Sch91]. After the first flip, due to Hassett and Hyeon, the model becomes an Artin stack with infinite stabilizers ($\mathbb{G}_m$'s). The second flip has been studied by J. Alper, D. Smyth, F. van der Wyck and M. Fedorchuk [ASW13].

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1. Quasi-projective stacks

In this section we give a brief account of the proof of Kresch's Theorem A, cf. [Kre09].

Note that the implications (iii) $\implies$ (ii) $\implies$ (i) in Theorem A follow immediately from the fact that projectivity (resp. quasi-projectivity) of stacks is preserved under closed (resp. open) immersions.

It is thus enough to show that every quasi-projective Deligne–Mumford stack $\mathscr{X}$ over a field of characteristic zero can be embedded in a smooth projective Deligne–Mumford stack $\mathscr{P}$ [Kre09, Thm. 5.3]. This is accomplished via GIT and Kirwan's theorem on the elimination of the strictly semistable locus [Kir85]. Let us outline the argument.

First, note that $\mathscr{X} = [Y/\mathrm{GL}_n]$ where Y is quasi-projective. Then, for some N, we may embed $Y \hookrightarrow \mathbb{P}^N$ equivariantly and such that GL_n acts linearly on $\mathbb{P}^N$ with Y contained in the stable locus $\left(\mathbb{P}^N\right)^s$. Finally, by Kirwan's result, there is an equivariant blow-up sequence with smooth centers $P \to \mathbb{P}^N$ that is an isomorphism over the stable locus and resolves the strictly semistable locus. This means that $P^{ss} = P^s$ so that $P^{ss}/\!/\mathrm{GL}_n = P^s/\mathrm{GL}_n$ is projective and there is an immersion $Y \hookrightarrow P^s$. The Deligne–Mumford stack $\mathscr{P} = [P^s/\mathrm{GL}_n]$ is thus smooth and projective and there is an immersion $\mathscr{X} \hookrightarrow \mathscr{P}$ as requested.

Remark **(1.1).** In the proof of Theorem A, the stack $\mathscr{P}$ is by construction a global quotient stack. To make the discussion complete, let us also mention that every separated Deligne–Mumford stack that is smooth over a field of characteristic zero with (quasi-)projective coarse moduli space is a global quotient stack and hence (quasi-)projective [Kre09, Thm. 4.4]. In the orbifold case, this is [EHKV01, Thm. 2.18]. In the general case, this is [KV04, Thm. 2] using Gabber's theorem that the Brauer group and the cohomological Brauer group coincide for quasi-projective schemes, cf. [Jon03].

2. Normal global quotient stacks

In this section, we give a proof of Theorem B. Both the theorem and its proof are due to D. Edidin (unpublished). The proof is quite similar to J.-C. Raoult's proof of the existence of compactifications of normal algebraic spaces [Rao71].

Let S be an algebraic space. Let $\mathscr{X}$ be a stack that is separated and of finite type over S. At the end of the proof we will assume that S is excellent to ensure that the normalized compactification is of finite type over S. For general S, and under the assumption that $\mathscr{X}$ has a finite number of irreducible components, the proof gives a compactification $\overline{\mathscr{X}}$ which is an E-stack but not necessarily normal.

Firstly, assume that $\mathscr{X}$ is a normal irreducible E-stack and apply the following lemma.

Lemma (2.1). *Let $\mathscr{X}$ be a normal irreducible E-stack. Then there exists a dense open substack $\mathscr{U} \subseteq \mathscr{X}$ and a finite morphism $p\colon Z \to \mathscr{X}$ such that Z is a normal irreducible scheme and $p|_{\mathscr{U}}$ is a G-torsor for a finite constant group G.*

Proof. By definition, there is a finite morphism $p\colon Z \to \mathscr{X}$ where Z is an algebraic space and a dense open substack $\mathscr{U} \subseteq \mathscr{X}$ such that $p|_{\mathscr{U}}$ is étale. After applying [LMB00, Thm. 16.6] (or [Ryd14, Thm. B] if $\mathscr{X}$ is not noetherian) to Z, we may assume that Z is a scheme. Let d be the rank of

p over $\mathscr{U}$. After replacing Z with the "Galois closure" $\mathrm{SEC}^d(Z/\mathscr{X})$ we may assume that $Z \to \mathscr{X}$ is a $G = \mathfrak{S}_d$-torsor over $\mathscr{U}$. After replacing Z with an irreducible component and G with a subgroup, we may assume that Z is irreducible. Finally, we may replace Z with its normalization in $p^{-1}(\mathscr{U})$. Note that the normalization is finite since $\mathscr{X}$ is normal and $p|_{\mathscr{U}}$ is étale [Bou64, Chap. V, §1, No. 6, Cor. 1 of Prop. 20]. □

Secondly, compactify Z:

Lemma (2.2). *Let Z be a normal irreducible scheme, separated and of finite type over S, with an action of a finite constant group G. Then there exists an algebraic space $\overline{Z}$ that carries an action of G, is proper over S and contains Z as an equivariant open subscheme. If S is excellent, then there is a* normal *algebraic space $\overline{Z}$ with the same properties.*

Proof. First we deal with the issue that S is an algebraic space and not a scheme. This is also the reason that $\overline{Z}$ is not necessarily a scheme.

We may assume that S is normal and irreducible. Applying Lemma (2.1), we obtain a normal *scheme* S' with an action of a constant group H and a finite ramified H-covering $S' \to S$. Since S is normal, this implies that $S = S'/H$ where S'/H denotes the coarse quotient. Let $Z' = Z \times_S S'$ which comes with an action of $G \times H$. After replacing Z' with its normalization, we may assume that Z' is normal, that $Z' \to Z$ is finite and that $Z = Z'/H$.

Now, take a compactification $Z' \subset \overline{Z'}$ as exists by Nagata's compactification theorem for schemes [Con07]. We can make this compactification $G \times H$-equivariant by the following standard trick. The fiber product $(\overline{Z'}/S)^{G\times H} = \overline{Z'} \times_S \cdots \times_S \overline{Z'}$ comes with a natural action of $G \times H$ and we may embed Z' in the fiber product equivariantly via $z' \mapsto \big(\sigma(z')\big)_{\sigma\in G\times H}$. We then replace $\overline{Z'}$ with the closure of Z' in the fiber product.

Finally, we let $\overline{Z} = \overline{Z'}/H$ be the coarse quotient, which exists by a theorem of Deligne, cf. [Mat76, Ryd13]. By the universality of coarse quotients, it comes equipped with a G-action.

If S is excellent, we may further replace $\overline{Z'}$ with its normalization. □

Thirdly, also assume that $\mathscr{X}$ is a global quotient stack. Then, by definition, there exists a vector bundle $\mathcal{E}$ on $\mathscr{X}$ such that the total space U of the corresponding frame bundle $U \to \mathscr{X}$ is an algebraic space.

Lemma (2.3). *For a suitable choice of $\overline{Z}$, the G-equivariant vector bundle $\mathcal{E}_Z = p^*\mathcal{E}$ extends to a G-equivariant vector bundle $\mathcal{E}_{\overline{Z}}$ on $\overline{Z}$.*

Proof. Consider $\overline{\mathscr{Y}} = [\overline{Z}/G]$ and $\mathscr{Y} = [Z/G]$ with the induced morphism $q\colon \mathscr{Y} \to \mathscr{X}$. We may extend $q^*\mathcal{E}$ to a finitely presented sheaf $\mathcal{E}_{\overline{\mathscr{Y}}}$ on $\overline{\mathscr{Y}}$ using [LMB00, Prop. 15.4] (or [Ryd14, Thm. A] if $\mathscr{X}$ is not noetherian). After a suitable blow-up $\widetilde{\mathscr{Y}} \to \overline{\mathscr{Y}}$, with center on the boundary $\overline{\mathscr{Y}} \setminus \mathscr{Y}$, the strict transform $\mathcal{E}_{\widetilde{\mathscr{Y}}}$ of $\mathcal{E}_{\overline{\mathscr{Y}}}$ becomes a vector bundle. This center is given by a Fitting ideal of $\mathcal{E}_{\overline{\mathscr{Y}}}$, see [RG71, Lem. 5.4.3] and [Ryd09, Prop. 4.10]. The pull-back of $\mathcal{E}_{\widetilde{\mathscr{Y}}}$ to $\widetilde{Z} = \overline{Z} \times_{\overline{\mathscr{Y}}} \widetilde{\mathscr{Y}}$ is a G-equivariant vector bundle extending $\mathcal{E}_Z$ as requested. □

We may now finish the proof of Theorem B. Let W and $\overline{W}$ denote the frame bundles of $\mathcal{E}_Z = p^*\mathcal{E}$ and the extension $\mathcal{E}_{\overline{Z}}$. Then, by construction, $W \to U$ is a finite ramified G-covering. Since U is a normal algebraic spaces, it follows that $U = W/G$ (the coarse quotient). We let $\overline{U} = \overline{W}/G$. By construction, $\overline{W}$ has an action by $G \times \mathrm{GL}_n$ so $\overline{U}$ comes with an action of GL_n. We let $\overline{\mathscr{X}} = [\overline{U}/\mathrm{GL}_n]$.

The finite morphism $\overline{W} \to \overline{U}$ induces a finite morphism $\overline{Z} = [\overline{W}/\mathrm{GL}_n] \to \overline{\mathscr{X}} = [\overline{U}/\mathrm{GL}_n]$. It follows that $\overline{\mathscr{X}}$ is proper over S. Since the finite morphism is étale over $\mathscr{U}$, it also follows that $\overline{\mathscr{X}}$ is an E-stack.

Remark **(2.4).** Theorem B also gives compactifications of morphisms as follows. Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a separated morphism between normal global quotient E-stacks of finite type over an algebraic space S. First choose compactifications $\mathscr{X} \subseteq \overline{\mathscr{X}}$ and $\mathscr{Y} \subseteq \overline{\mathscr{Y}}$. The induced morphism $\varphi\colon \mathscr{X} \to \overline{\mathscr{X}} \times_S \overline{\mathscr{Y}}$ is quasi-finite, representable and separated. Using Zariski's Main Theorem, we obtain a factorization $\mathscr{X} \subseteq \widetilde{\mathscr{X}} \to \overline{\mathscr{X}} \times_S \overline{\mathscr{Y}}$ where the first morphism is an open immersion and the second is finite. The resulting morphism $\overline{f}\colon \widetilde{\mathscr{X}} \to \overline{\mathscr{Y}}$ is proper and compactifies f. To get a compactification of $\mathscr{X}$ over $\mathscr{Y}$, it suffices to take $\overline{f}|_{\mathscr{Y}}\colon \overline{f}^{-1}(\mathscr{Y}) \to \mathscr{Y}$.

3. Tame Deligne–Mumford stacks

In this section, we give a rough outline of the proof of Theorem C which is much more involved than the two previous proofs. We refer to [Ryd09] for all details. The main tool behind Theorem C is a new class of stacky modifications—tame stacky blow-ups—and an "étalification" theorem which in its simplest form asserts that a tamely ramified finite flat cover becomes étale after a tame stacky blow-up. This is analogous to Raynaud–Gruson's flatification by blow-ups theorem.

Let $f\colon \mathscr{X} \to \mathscr{Y}$ be a tame morphism of Deligne–Mumford stacks. We are looking for a proper tame morphism $\overline{f}\colon \overline{\mathscr{X}} \to \mathscr{Y}$ that compactifies f. The proof that such a compactification exists goes through the following steps:

(i) Show that the existence of a compactification is étale-local on $\mathscr{Y}$. In particular, we may assume that $\mathscr{Y} = Y$ is a scheme.

(ii) Consider the coarse space X of $\mathscr{X}$ and choose a compactification $\overline{X}$ over Y. It is enough to compactify $\mathscr{X} \to \overline{X}$ so we may replace Y with $\overline{X}$. Using step (i) once again, we may assume that $\overline{X}$ is a scheme.

(iii) Show that, after modifying the boundary $\overline{X} \setminus X$ by an X-modification $\widetilde{X} \to \overline{X}$, there is a compactification étale-locally on $\widetilde{X}$. By step (i), there is then a compactification $\overline{\mathscr{X}} \to \widetilde{X} \to \overline{X}$ of $\mathscr{X} \to \widetilde{X} \to \overline{X}$.

As we will see, it is in step (i) that the real difficulties materialize and where stacky blow-ups and the étalification theorem are crucial.

3.1. **Compactifying algebraic spaces.** Step (ii) requires the existence of compactifications of algebraic spaces (over a scheme). This has been proved by Raoult [Rao74] under the assumption that the base is an excellent scheme. Using [Ryd14], this assumption is easily removed, cf. [CLO12] or [Ryd09, Thm. B and §6] which also treats representable morphisms between stacks.

3.2. **Finding a compactification étale-proper-locally on the coarse space.** Step (iii) is more involved. We know that étale-locally on X, we can write $\mathscr{X} = [U/G]$ for some finite constant group G. Stacks with such quotient presentations are easy to compactify, cf. Lemma (2.2). The problem is that étale-locally on X is not the same thing as étale-locally on $\overline{X}$. Given an étale surjection $U \to X$, we could try to use Zariski's main theorem to obtain a finite surjection $\overline{U} \to \overline{X}$. In general, however, the extension to $\overline{X}$ would be ramified along the boundary. Perhaps even more seriously, if $U \to X$ is not finite, then $\overline{U} \to \overline{X}$ need not even be étale over X.

We overcome this problem by first modifying the boundary of $\overline{X}$. This is accomplished using *Riemann–Zariski spaces.* The Riemann–Zariski space RZ of $\overline{X}$ is the limit of all modifications of $\overline{X}$ and the local rings of RZ are valuation rings dominating $\overline{X}$. To cover the Riemann–Zariski space, it is crucial to also treat non-discrete valuations and hence to deal with non-noetherian rings.

The idea is now as follows. Assume that we can compactify $\mathscr{X} \to \overline{X}$ after the base change to the spectra of any valuation ring dominating $\overline{X}$. An easy limit argument and the quasi-compactness of the Riemann–Zariski space then show that there exists a proper birational morphism $\widetilde{X} \to \overline{X}$ and an open covering $\widetilde{X}' \to \widetilde{X}$ such that $\mathscr{X} \to \overline{X}$ can be compactified after the base change along $\widetilde{X}' \to \widetilde{X} \to \overline{X}$.

Since we do not want to modify X, we need a slightly more refined version of this argument. Thus, consider instead the *relative Riemann–Zariski space* to the pair $(\overline{X}, X)$ [Tem11]. This space is the limit of all X-admissible modifications of $\overline{X}$. In this way, we reduce to the case where $\overline{X}$ is the spectrum W of a local ring of the relative Riemann–Zariski space.

These local rings are not valuation rings but merely *semi-valuation rings.* In our situation, the spectrum W of such a local ring is the pushout of a diagram

$$\operatorname{Spec} V \hookleftarrow \operatorname{Spec} \kappa(x) \hookrightarrow \operatorname{Spec}(\mathcal{O}_{X,x})$$

where $V \subseteq \kappa(x)$ is a valuation ring centered on $\overline{X}$. Note that the generic point of $\operatorname{Spec} V$ is glued to the closed point of $\operatorname{Spec}(\mathcal{O}_{X,x})$. Moreover, $\operatorname{Spec}(\mathcal{O}_{X,x})$ is an open subscheme of W and $\operatorname{Spec} V$ is a closed subscheme of W.

Assume now that $\overline{X} = W$ since it is enough to deal with this situation. As mentioned before, we know that $\mathscr{X} = [U/G]$ étale-locally over X but we want such a presentation étale-locally over $\overline{X}$. This can be obtained in two different ways:

(a) Over a semi-valuation ring, any étale morphism $X' \to X$ with trivial residue field extension over x automatically extends to an étale morphism $\overline{X'} \to \overline{X}$. We can write $\mathscr{X} = [U/G]$ after such an étale morphism, although we cannot guarantee that G is tame even if $\mathscr{X}$ is tame.

(b) Use tame étalification to extend any étale morphism over X that is tamely ramified over the boundary $\overline{X} \setminus X$ to an étale morphism over $\overline{X}$ after a stacky modification of $(\overline{X}, X)$. In this way we can arrange so that G is tame.

The first approach is sufficient in characteristic zero and the second approach is sufficient in positive and mixed characteristics.

When we have accomplished that $\mathscr{X} = [U/G]$, we can pick a G-equivariant compactification of U over $\overline{X}$. This is elementary: $U \to X$ is finite so there is a compactification $\overline{U} \to \overline{X}$ by Zariski's main theorem. After replacing $\overline{U}$ with a closure of U in a fiber product, as in the proof of Lemma (2.2), we obtain a G-equivariant compactification $\overline{U}$. Finally, we let $\overline{\mathscr{X}} = [\overline{U}/G]$. Since G is tame, so is $\overline{\mathscr{X}}$.

3.3. **Gluing étale-local compactifications.** It remains to settle step (i). Let $g\colon \overline{X}' \to \overline{X}$ be an étale morphism, let $X' = g^{-1}(X)$ and let $\mathscr{X}' = \mathscr{X} \times_X X'$. Given a compactification $\overline{\mathscr{X}}' \to \overline{X}'$ of $\mathscr{X}' \to \overline{X}'$, we are looking for a procedure to descend this to a compactification of $\mathscr{X} \to \overline{X}$. The problem is that since the compactification is not canonical, it does not come with a descent datum. Using étale dévissage [Ryd11], we further reduce the problem to the following two cases:

(a) g is finite étale.
(b) g is an étale neighborhood of a closed subscheme $Z \subseteq \overline{X}$ and the morphism $f\colon \mathscr{X} \to \overline{X}$ can be compactified after restricting to either $\overline{X}'$ or $U = \overline{X} \setminus Z$.

In the first case, a compactification is easily obtained by taking the Weil restriction of $\overline{\mathscr{X}}' \to \overline{X}'$ along g. The second case is where the essence of Theorem C is revealed. To avoid further technicalities, let us restrict the discussion to the slightly simpler case where $\overline{X}' = U_1 \amalg U_2$ is an open covering of $\overline{X}$ and $Z = U_2 \setminus U_1$. Then, by assumption, we know that there exists compactifications of $f\colon \mathscr{X} \to \overline{X}$ after restricting to either U_1 or U_2. However, these two compactions need not agree over $U_1 \cap U_2$. If we instead would glue the compactifications along $\mathscr{X}|_{U_1 \cap U_2}$, then the result would be non-separated. This difficulty is at the core of the problem and the subject of the next subsection.

3.4. **Stacky modifications.** Given two compactifications $\mathscr{X} \subset \overline{\mathscr{X}}_1$ and $\mathscr{X} \subset \overline{\mathscr{X}}_2$, there is a third compactification that dominates both. This is quite straight-forward: the product $\overline{\mathscr{X}}_1 \times \overline{\mathscr{X}}_2$ is proper over the base and there is a canonical morphism $j\colon \mathscr{X} \to \overline{\mathscr{X}}_1 \times \overline{\mathscr{X}}_2$. Since we are dealing with stacks, the morphism j is not an immersion but it is at least quasi-finite, separated and representable. Thus, using Zariski's main theorem, we obtain a factorization $\mathscr{X} \subset \overline{\mathscr{X}} \to \overline{\mathscr{X}}_1 \times \overline{\mathscr{X}}_2$ where the first morphism is an open immersion and the second is finite. We have thus obtained a compactification $\mathscr{X} \subset \overline{\mathscr{X}}$ that dominates $\overline{\mathscr{X}}_1$ and $\overline{\mathscr{X}}_2$.

Now, in the setting of the previous subsection, we were starting with two compactifications $\mathscr{X}|_{U_1} \subset \overline{\mathscr{X}}_1$ and $\mathscr{X}|_{U_2} \subset \overline{\mathscr{X}}_2$ over U_1 and U_2 respectively. Arguing as above, we can find a compactification $\mathscr{X}|_{U_1 \cap U_2} \subset \overline{\mathscr{X}}_{12}$ over $U_1 \cap U_2$ that dominates $\overline{\mathscr{X}}_1|_{U_1 \cap U_2}$ and $\overline{\mathscr{X}}_2|_{U_1 \cap U_2}$. The next step is to extend this compactification to U_1 and U_2 respectively. This is closely related to the following special case of the compactification theorem.

Problem **(3.1).** Let $\mathscr{Y}$ be a Deligne–Mumford stack and let $\mathscr{U} \subset \mathscr{Y}$ and $\mathscr{V} \subset \mathscr{Y}$ be open substacks. Further, let $\mathscr{U}' \to \mathscr{U}$ be a proper morphism that is an isomorphism over $\mathscr{U} \cap \mathscr{V}$. Can we then extend $\mathscr{U}' \to \mathscr{U}$ to a proper morphism $\mathscr{Y}' \to \mathscr{Y}$ that is an isomorphism over $\mathscr{V}$? That is, can we compactify $\mathscr{U}' \cup \mathscr{V}$ relative to $\mathscr{Y}$?

In our setting, we would like to apply this to $\mathscr{Y} = \overline{\mathscr{X}}_1$, $\mathscr{V} = \mathscr{X}|_{U_1}$, $\mathscr{U} = \overline{\mathscr{X}}_1|_{U_1 \cap U_2}$ and $\mathscr{U}' = \overline{\mathscr{X}}_{12}$. Note that the problem is formulated in such a way that the base $\overline{X}$ does not play a role any longer.

Problem (3.1) is about *stacky modifications*, that is, proper not necessarily representable morphisms that are isomorphisms over an open subset. The general case of Problem (3.1) appears to be about as difficult as Theorem C. The trick is to only use Problem (3.1) for a class of explicit stacky modifications $\mathscr{U}' \to \mathscr{U}$ for which the problem is easy by definition.

For usual modifications, one such class is *blow-ups*. If $\mathscr{U}' \to \mathscr{U}$ is a blow-up in the center $\mathscr{Z}$, then we can let $\mathscr{Y}' \to \mathscr{Y}$ be the blow-up in the center $\overline{\mathscr{Z}}$: the closure of $\mathscr{Z}$ in $\mathscr{Y}$.

For tame stacky modifications, there is a similar class called *tame stacky blow-ups*. These are compositions of *root stacks* and blow-ups, both of which can easily be extended along open immersions in the same way as blow-ups. The problem is thus reduced to finding a compactification $\overline{\mathscr{X}}_{12}$ that dominates $\overline{\mathscr{X}}_1|_{U_1 \cap U_2}$ and $\overline{\mathscr{X}}_2|_{U_1 \cap U_2}$ such that the morphisms $\overline{\mathscr{X}}_{12} \to \overline{\mathscr{X}}_1|_{U_1 \cap U_2}$ and $\overline{\mathscr{X}}_{12} \to \overline{\mathscr{X}}_2|_{U_1 \cap U_2}$ are stacky modifications. This follows from the following key result of [Ryd09].

Theorem (3.2) ([Ryd09, Cor. E]). *Let $\mathscr{Y}$ be a Deligne–Mumford stack. Let $\mathscr{U} \subseteq \mathscr{Y}$ be an open substack and let $\mathscr{Y}' \to \mathscr{Y}$ be a $\mathscr{U}$-admissible tame stacky modification. Then there exists a $\mathscr{U}$-admissible tame stacky blow-up $\mathscr{Y}'' \to \mathscr{Y}'$ such that the composition $\mathscr{Y}'' \to \mathscr{Y}' \to \mathscr{Y}$ is a tame stacky blow-up.*

As compositions of tame stacky blow-ups are tame stacky blow-ups, we can apply the theorem twice to obtain a compactification $\overline{\mathscr{X}}_{12}$ such that the morphisms $\overline{\mathscr{X}}_{12} \to \overline{\mathscr{X}}_1|_{U_1 \cap U_2}$ and $\overline{\mathscr{X}}_{12} \to \overline{\mathscr{X}}_2|_{U_1 \cap U_2}$ are tame stacky blow-ups. We may then extend these to tame stacky blow-ups $\widetilde{\mathscr{X}_1} \to \overline{\mathscr{X}}_1$ and $\widetilde{\mathscr{X}_2} \to \overline{\mathscr{X}}_2$ which then glues to a compactification of $\mathscr{X}$ over $\overline{X} = U_1 \cup U_2$.

The analogue of Theorem (3.2) for usual modifications and blow-ups is a key result needed for Nagata's compactification of schemes. In fact, this result occupies a large part of the proof. It is, however, also a direct consequence of Raynaud–Gruson's flatification by blow-up theorem [RG71, Thm. 5.2.2] since a flat modification is an isomorphism. For Theorem (3.2), Raynaud–Gruson's result is not enough since a flat stacky modification need not be an isomorphism. The prominent example is a root stack, which is a non-trivial flat stacky modification. On the other hand, an *étale* stacky modification is an isomorphism. Therefore, Theorem (3.2) is an immediate corollary of the following étalification theorem.

Theorem (3.3) (Tame étalification by tame stacky blow-ups [Ryd09, Thm. C]). *Let $\mathscr{Y}$ be a Deligne–Mumford stack. Let $\mathscr{U} \subset \mathscr{Y}$ be an open substack and let $\mathscr{X} \to \mathscr{Y}$ be a morphism of finite type such that $\mathscr{X}|_{\mathscr{U}} \to \mathscr{U}$ is étale and* *tamely ramified along the boundary. Then there exists a $\mathscr{U}$-admissible tame stacky blow-up $\widetilde{\mathscr{Y}} \to \mathscr{Y}$ and a $\mathscr{X}|_{\mathscr{U}}$-admissible blow-up $\widetilde{\mathscr{X}} \to \mathscr{X} \times_{\mathscr{Y}} \widetilde{\mathscr{Y}}$ such that $\widetilde{\mathscr{X}} \to \widetilde{\mathscr{Y}}$ is étale.*

The tame ramification assumption on $\mathscr{X} \to \mathscr{Y}$ is automatically fulfilled if the morphism is tame, in the sense that the automorphism groups have order prime to p, and $\mathscr{X}|_{\mathscr{U}} \to \mathscr{U}$ is finite étale of rank prime to p. In the application above, this is satisfied since $\mathscr{X}|_{\mathscr{U}} \to \mathscr{U}$ is an isomorphism.

The proof of the tame étalification theorem takes up the bulk of [Ryd09]. First of all one needs to establish some general properties of tame stacky blow-ups similar to those of blow-ups. The proof of the étalification theorem is then accomplished using relative Riemann–Zariski spaces to reduce to the case of a semi-valuation ring. This case is handled by well-established tame ramification theory.

4. Non-uniformizable stacks and extension of stackiness across the boundary

In this section we prove Theorem D, that is, we give an example of a stack $\mathscr{U}$ with coarse space U and a compactification $U \subseteq X$ such that there is no compactification $\mathscr{U} \subseteq \mathscr{X}$ with coarse space X. The counter-example is based on the notion of uniformizable stacks.

Definition (4.1) ([Noo04])**.** Let $\mathscr{X}$ be a Deligne–Mumford stack. We say that $\mathscr{X}$ is *uniformizable* if there exists a finite étale cover $U \to \mathscr{X}$ such that U is an algebraic space. We say that $\mathscr{X}$ is (algebraically) *simply connected* if every finite étale cover $\mathscr{X}' \to \mathscr{X}$ is trivial, i.e., every connected component of $\mathscr{X}'$ is isomorphic to its image.

The idea is now very simple. As is well-known, the stack $\mathscr{U}$ is uniformizable étale-locally on U. If $\mathscr{X}$ would exist, then $\mathscr{U}$ would be uniformizable étale-locally on X. It is thus enough to find $\mathscr{U}$ and $U \subseteq X$ such that $\mathscr{U} \times_X \operatorname{Spec}(\mathcal{O}_{X,x}^{\mathrm{sh}})$ is simply connected and not an algebraic space.

The example will be constructed using root stacks and root gerbes. For a definition of these, we refer to C. Cadman's paper [Cad07].

Proposition (4.2). *Let X be a scheme.*

- (i) *Let $D \hookrightarrow X$ be an effective Cartier divisor such that $\mathcal{O}(D)$ is trivial. Then the root stack $X(\sqrt[r]{D})$ is uniformizable.*
- (ii) *Let $\mathcal{L}$ be an invertible sheaf. Then the root gerbe $X(\sqrt[r]{\mathcal{L}^r})$ is neutral and hence uniformizable.*

Proof. If $\mathcal{O}(D)$ is trivial, then we can construct a cyclic covering $X' \to X$ of degree r, ramified along D. Explicitly, $X' = \operatorname{Spec}_X(\mathcal{O}_X[t]/t^r - s)$ where s is a global section of $\mathcal{O}(D) = \mathcal{O}_X$ defining D. The induced morphism $X' \to X(\sqrt[r]{D})$ is finite and étale. The second statement is obvious from the definition. ☐

We give an example where $\mathcal{O}(D)$ is non-trivial but torsion and the root stacks still are uniformizable.

Example (4.3). Let k be a field of characteristic not equal to 2. Consider the action of $\mathfrak{S}_2$ on $X' = \mathbb{A}^2 = \mathrm{Spec}(k[x,y])$ that takes $(x,y) \to (-x,-y)$. Let $X = X'/\mathfrak{S}_2 = \mathrm{Spec}(k[x^2,xy,y^2])$, let $X_0 = X \setminus \{0\}$ be the smooth locus of X and let $X_0' = X' \setminus \{0\}$ be the inverse image of X_0. The Weil divisor $D = \mathrm{V}(x^2,xy) \hookrightarrow X$ is $\mathbb{Q}$-Cartier. Let $D_0 = D \cap X_0$ be the restriction to the smooth locus and let $D_0' = \mathrm{V}(x)$ be the inverse image in X_0'. Let $r \geq 2$ be an integer. There is a finite étale morphism $X_0'(\sqrt[r]{D_0'}) \to X_0(\sqrt[r]{D_0})$. The stack $X_0'(\sqrt[r]{D_0'})$ is uniformizable and hence so is $X_0(\sqrt[r]{D_0})$. Note that $\mathcal{O}(D_0)$ is non-trivial since X is normal and D is not Cartier. On the other hand, $2D = \mathrm{V}(x^2)$ is Cartier and $\mathcal{O}(2D)$ is trivial.

For a converse statement we have:

Proposition (4.4). *Let k be a field (resp. a separably closed field) of characteristic zero. Let $P \in \mathbb{P}^1_k$ be a k-rational point and let $r \geq 2$ be an integer. Then the stacks $\mathbb{P}^1(\sqrt[r]{P})$ and $\mathbb{P}^1(\sqrt[r]{\mathcal{O}(1)})$ are not uniformizable (resp. are simply connected).*

Proof. It is enough to prove that the stacks are simply connected when k is separably closed. But $\mathbb{P}^1(\sqrt[r]{P})$ and $\mathbb{P}^1(\sqrt[r]{\mathcal{O}(1)})$ equals the weighted projective stacks $\mathbb{P}(1,r)$ and $\mathbb{P}(r,r)$, respectively, which are simply connected. To see this, let $X \to \mathbb{P}^1(\sqrt[r]{P})$ be the $\mathbb{G}_m$-torsor corresponding to $\mathcal{O}(P^{1/r})$. Explicitly, we have

$$\mathbb{P}^1 = \mathrm{Spec}(k[x,y]) \setminus \{0\}/\mathbb{G}_m, \quad P = \mathrm{V}_+(f),\ f \in k[x,y]_1,$$

$$X = \mathrm{Spec}(A[x,y,z]/z^r - f) \setminus \{0\}$$

where $\mathbb{G}_m$ acts on X with weight r on x and y, and weight 1 on z. The scheme X sits inside $\mathbb{A}^2$ with complement of codimension 2 so X is simply connected. Since $X \to \mathbb{P}^1(\sqrt[r]{P})$ has geometrically connected fibers, it follows that $\mathbb{P}^1(\sqrt[r]{P})$ is simply connected. Similarly, the tautological $\mathbb{G}_m$-torsor on $\mathbb{P}^1(\sqrt[r]{\mathcal{O}(1)})$ has total space $X = \mathrm{Spec}(A[x,y]) \setminus \{0\}$ where x and y have weight r. □

In positive characteristic, projective space is simply connected and it is plausible that weighted projective stacks are simply connected as well. However, the simple argument above fails as affine space is not simply connected in positive characteristic.

Theorem (4.5). *Let S be the spectrum of a henselian local ring with closed point s. Let X be an algebraic stack that is proper over S with finite diagonal. The functor*

$$\{\textit{Finite étale covers of } X\} \longrightarrow \{\textit{Finite étale covers of } X_s\}$$
$$(E \to X) \longmapsto (E_s \to X_s)$$

is an equivalence of categories.

Proof. The theorem is well-known when X is a scheme (cf. [EGA$_{\mathrm{IV}}$, Thm. 18.3.4] for the case where S is noetherian and complete and [Art69, Thm. 3.1] or [SGA$_4$, Exp. XII, Thm. 5.9 bis] for the henselian case). To prove the case where X is a stack, choose a finite (non-flat) surjection $Z \to X$ [Ryd14, Thm. B]. By descent along finite surjections, there is an equivalence between the category of finite étale covers of X (resp. X_s) and the category of finite étale covers of Z (resp. Z_s) together with a descent datum on $Z \times_X Z$ (resp. $(Z \times_X Z)_s$), cf. [SGA$_1$, Exp. VI, Thm. 4.7] or [Ryd10]. From the case of schemes, the category of finite étale covers of Z equipped with a descent datum is equivalent to the category of finite étale covers of Z_s equipped with a descent datum. □

Theorem (4.6). *Let k be a field of characteristic zero and let $X \hookrightarrow \mathbb{A}^4$ be the hypersurface* $\operatorname{Spec}(k[x,y,z,w]/xy-zw)$. *Let $X_0 = X \setminus \{0\}$ be the smooth locus of X and let $D_0 \hookrightarrow X_0$ be the Cartier divisor given by $x = z = 0$. Let $\mathscr{X}_0$ be either the orbifold $X_0(\sqrt[r]{D_0})$ or the gerbe $X_0(\sqrt[r]{\mathcal{O}(D_0)})$ so that X_0 is the coarse moduli space of $\mathscr{X}_0$. Then there does not exists a stack $\mathscr{X}$ with finite diagonal and coarse moduli space X such that $\mathscr{X}|_{X_0} \cong \mathscr{X}_0$.*

The scheme X is a non-simplicial toric variety and hence normal but the singular point is not a quotient singularity. The Picard group of X_0 is $\mathbb{Z}$ and is generated by D_0 and $(X_0)_{\overline{k}}$ is simply connected.

Proof of Theorem (4.6). We can assume that k is separably closed. Let W denote the spectrum of the (strict) henselization of X at the singular point $x = y = z = w = 0$. If a stack $\mathscr{X}$ as in the Theorem exists, then $\mathscr{X} \times_X W$, and *a fortiori* $\mathscr{X}_0 \times_X W$, are uniformizable.

Let $X' \to X$ be the small resolution such that the closure of D_0 intersects the exceptional fiber in one point. Explicitly

$$X' = \big(\operatorname{Spec}(k[u_1,u_2,u_3,u_4]) \setminus \{u_1 = u_2 = 0\}\big)/\mathbb{G}_m$$

where $\mathbb{G}_m$ acts freely with weights $(1,1,-1,-1)$ and $x = u_1u_3$, $y = u_2u_4$, $z = u_1u_4$ and $w = u_2u_3$. The closure D' of D_0 in X' is given by $u_1 = 0$, the exceptional fiber $E \cong \mathbb{P}^1 \hookrightarrow X'$ is defined by $u_3 = u_4 = 0$ and $D'|_E$ has degree one. Let $\mathscr{X}' = X'(\sqrt[r]{D'})$ or $\mathscr{X}' = X'(\sqrt[r]{\mathcal{O}(D')})$.

The finite étale covers of $\mathscr{X}_0 \times_X W$ coincide with the finite étale covers of $\mathscr{X}' \times_X W$ since $\mathscr{X}_0 \subseteq \mathscr{X}'$ has complement of codimension 2 and $\mathscr{X}'$ is regular (this is why we study the small resolution of X). Since $\mathscr{X}' \to X$ is proper and W is henselian, the finite étale covers of $\mathscr{X}' \times_X W$ are the same as the finite étale covers of $\mathscr{X}'|_0 \cong \mathbb{P}^1(\sqrt[r]{P}) \cong \mathbb{P}(1,r)$ or $\mathscr{X}'|_0 \cong \mathbb{P}^1(\sqrt[r]{\mathcal{O}(1)}) \cong \mathbb{P}(r,r)$ by Theorem (4.5) but these stacks are simply connected as we saw in Proposition (4.4). Thus, $\mathscr{X}_0 \times_X W$ is simply connected and hence not uniformizable. This contradicts the existence of $\mathscr{X}$. □

Remark **(4.7).** The stack $\mathscr{X}_0$ of the theorem has an *Artin compactification*, i.e., there is a stack $\mathscr{X}$ with *good moduli space* X [Alp13]. This is the stack $[\mathbb{A}^4(\sqrt[r]{u_1 = 0})/\mathbb{G}_m]$. The moduli map $\mathscr{X} \to X$ is universally closed but not separated.

The scheme X has a terminal (but not $\mathbb{Q}$-factorial) singularity. This indicates that there is a smooth DM-stack such that when running the minimal model program on this stack, we are supposed to do a contraction which exists on the level of coarse moduli spaces but not on the level of stacks, not even as a rational map. A solution to this could be to instead work in the category of Artin stacks. This has some twists to it, for example an open immersion of Artin stacks could act as a contraction!

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