ARTIN'S CRITERIA FOR ALGEBRAICITY REVISITED

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Abstract. Using notions of homogeneity, as developed in [Hal12b], we give new proofs of M. Artin's algebraicity criteria for functors [Art69b, Thm. 5.3] and groupoids [Art74, Thm. 5.3]. Our methods give a more general result, unifying Artin's two theorems and clarifying their differences. In positive characteristic, we also identify a subtle issue in Artin's algebraicity criterion for stacks. With the techniques that we develop, this problem is circumvented.

Introduction

Classically, moduli spaces in algebraic geometry are constructed using either projective methods or by forming suitable quotients. In his reshaping of the foundations of algebraic geometry half a century ago, Grothendieck shifted focus to the functor of points and the central question became whether certain functors are representable. Early on, he developed formal geometry and deformation theory, with the intent of using these as the main tools for proving representability. Grothendieck's proof of the existence of Hilbert and Picard schemes, however, is based on projective methods. It was not until ten years later that Artin completed Grothendieck's vision in a series of landmark papers. In particular, Artin vastly generalized Grothendieck's existence result and showed that the Hilbert and Picard schemes exist—as algebraic spaces—in great generality. It also became clear that the correct setting was that of algebraic spaces—not schemes—and algebraic stacks.

In his two eminent papers [Art69b, Art74], M. Artin gave precise criteria for algebraicity of functors and stacks. These criteria were later clarified and simplified by B. Conrad and J. de Jong [CJ02], who replaced Artin approximation with Néron–Popescu desingularization, by H. Flenner [Fle81] using Exal, and the first author [Hal12b] using coherent functors. The criterion in [Hal12b] is very streamlined and elegant and suffices—to the best knowledge of the authors—to deal with all present problems. It does not, however, supersede Artin's criteria as these are weaker. Another conundrum is the fact that Artin gives two different criteria—the first [Art69b, Thm. 5.3] is for functors and the second [Art74, Thm. 5.3] is for stacks—but neither completely generalizes the other.

The purpose of this paper is to use the ideas of Flenner and the first author to give a new criterion that supersedes all present criteria. We also introduce several new ideas that strengthen the criteria and simplify the proofs of [Art69b, Art74, Fle81]. We now state our criterion for algebraicity.

Main Theorem. *Fix an excellent, locally noetherian scheme S. Then, a category X, fibered in groupoids over the category of S-schemes,* $\mathbf{Sch}/S$*, is an algebraic*

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stack, locally of finite presentation over S, if and only if it satisfies the following conditions.

(1) [Stack] *X is a stack over $(\mathbf{Sch}/S)_{\mathrm{fppf}}$.*
(2) [Limit preserving] *For any cofiltered system of affine S-schemes $\{\operatorname{Spec} A_i\}_{i\in I}$ with limit $\operatorname{Spec} A$, the natural functor:*

$$\varinjlim_i X(\operatorname{Spec} A_i) \to X(\operatorname{Spec} A)$$

is an equivalence of categories.
(3) [Homogeneity] *For any diagram of local artinian S-schemes of finite type $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A']$, where $A' \twoheadrightarrow A$ is surjective and the residue field extension $B/\mathfrak{m}_B \to A/\mathfrak{m}_A$ is trivial, the natural functor:*

$$X(\operatorname{Spec}(A' \times_A B)) \to X(\operatorname{Spec} A') \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} B)$$

is an equivalence of categories.
(4) [Effectivity] *For any local noetherian ring $(B, \mathfrak{m})$, such that B is $\mathfrak{m}$-adically complete, with an S-scheme structure $\operatorname{Spec} B \to S$ such that the induced morphism $\operatorname{Spec}(B/\mathfrak{m}) \to S$ is locally of finite type, the natural functor:*

$$X(\operatorname{Spec} B) \to \varprojlim_n X(\operatorname{Spec}(B/\mathfrak{m}^n))$$

is an equivalence of categories.
(5) [Conditions on automorphisms and deformations] *For any affine, integral, and locally of finite type S-scheme $V_0 = \operatorname{Spec} R_0$ and $\xi_0 \in X(V_0)$:*
   (a) [Boundedness] *the following R_0-modules are coherent:*
      (i) $\operatorname{Aut}_{X/S}(\xi_0, \mathscr{O}_{V_0})$,
      (ii) $\operatorname{Def}_{X/S}(\xi_0, \mathscr{O}_{V_0})$;
   (b) [Constructibility] *there exists a dense open subset $U_0 \subset |V_0|$ such that for all $u \in U_0$, the following canonical maps are isomorphisms:*
      (i) $\operatorname{Aut}_{X/S}(\xi_0, \mathscr{O}_{V_0}) \otimes_{R_0} \kappa(u) \to \operatorname{Aut}_{X/S}(\xi_0, \kappa(u))$,
      (ii) $\operatorname{Def}_{X/S}(\xi_0, \mathscr{O}_{V_0}) \otimes_{R_0} \kappa(u) \to \operatorname{Def}_{X/S}(\xi_0, \kappa(u))$;
   (c) [Zariski localization] *for any open affine subscheme $U_0 \subset V_0$, and any $u \in |U_0|$ of finite type, the following canonical maps are isomorphisms:*
      (i) $\operatorname{Aut}_{X/S}(\xi_0, \kappa(u)) \to \operatorname{Aut}_{X/S}(\xi_0|_{U_0}, \kappa(u))$,
      (ii) $\operatorname{Def}_{X/S}(\xi_0, \kappa(u)) \to \operatorname{Def}_{X/S}(\xi_0|_{U_0}, \kappa(u))$.
(6) [Conditions on obstructions] *The 1-morphism $X \to \mathbf{Sch}/S$ satisfies:*
   (a) [Constructibility] *Condition 7.3 or 7.4;*
   (b) [Zariski localization] *Condition 7.6 or 7.7.*

In addition,

(i) *if S is Jacobson, e.g., of finite type over* $\operatorname{Spec}(\mathbb{Z})$, *then Conditions* (5c) *and* (6b) *can be omitted;*
(ii) *if S is a $\mathbb{Q}$-scheme, or we allow purely inseparable residue field extensions in* (3), *then it is enough that X is a stack over $(\mathbf{Sch}/S)_{\acute{\mathrm{E}}\mathrm{t}}$ in* (1).

All existing algebraicity proofs, including ours, consist of the following four steps:

(i) existence of formally versal deformations;
(ii) algebraization of formally versal deformations;
(iii) openness of formal versality; and
(iv) formal versality implies formal smoothness.

Step (i) was satisfactory dealt with by Schlessinger [Sch68, Thm. 2.11] for functors and Rim [SGA7, Exp. VI] for groupoids. This step uses conditions (3) and (5(a)ii). Step (ii) begins with the effectivization of formally versal deformations using condition (4). One may then algebraize this family using either Artin's results [Art69a, Art69b] or B. Conrad and J. de Jong's results [CJ02]. In the latter approach, Artin approximation is replaced with Néron–Popescu desingularization and S is only required to be excellent. This step requires condition (2).

The last two steps are more subtle and it is here that [Art69b, Art74, Fle81, Sta06, Hal12b] and our present treatment diverges—both when it comes to the criteria themselves and the techniques employed. We begin with discussing step (iv).

It is readily seen that our criterion is weaker than Artin's two criteria [Art69b, Art74] except that, in positive characteristic, we need X to be a stack in the fppf topology, or otherwise strengthen (3). This is similar to [Art69b, Thm. 5.3] where the functor is assumed to be an fppf-sheaf. In [*loc. cit.*], Artin uses the fppf sheaf condition to deduce that formally universal deformations are formally étale [Art69b, pp. 50–52], settling step (iv) for functors. In his second paper [Art74], Artin only assumes that the groupoid is an étale stack. His proof of step (iv) for groupoids [Art74, Prop. 4.2], however, does not treat inseparable extensions. We do not understand how this problem can be overcome, without strengthening the criteria and assuming that either the groupoid is a stack in the fppf topology or requiring homogeneity for inseparable extensions. Flenner does not discuss formal smoothness, and in [Hal12b] formal smoothness is obtained by strengthening the homogeneity condition (3).

With a completely different and simple argument, we show that formal versality and formal smoothness are equivalent. The idea is that with *homogeneity*, rather than *semi-homogeneity*, we can use the stack condition (1) to obtain homogeneity for artinian rings with arbitrary residue field extensions (Lemma 1.4). This immediately implies that formal versality and formal smoothness are equivalent (Lemma 2.3) so we accomplish step (iv) without using obstruction theories.

It is then not difficult to deduce, using step (ii), that we have homogeneity for arbitrary integral morphisms (Lemma 8.2). This also explains why less homogeneity is needed in [Art69b] than in [Art74]. We thus start with less homogeneity than in [Art69b], and prove that we have even more homogeneity than in [Art74].

Finally, Step (iii) uses constructibility, boundedness, and Zariski-localization of deformations and obstruction theories (Theorem 4.4). In our treatment, localization is only required when passing to non-closed points of finite type. Such points only exist when S is not Jacobson, e.g., if S is the spectrum of a discrete valuation ring. Our proof is very similar to Flenner's proof. It may appear that Flenner does not need Zariski localization in his criterion, but this is due to the fact that his conditions are expressed in terms of deformation and obstruction *sheaves*.

Outline. In Section 1, we recall the notions of homogeneity, limit preservation and extensions from [Hal12b]. We also introduce homogeneity that only involves artinian rings and show that residue field extensions are harmless for stacks in the fppf topology. In Section 2, we then relate formal versality, formal smoothness and vanishing of Exal.

In Section 3, we study additive functors and their vanishing loci. This is applied in Section 4 where we give conditions on Exal that assures that the locus of formal versality is open. The results are then assembled in Theorem 4.4.

In Section 5, we repeat the definitions of automorphisms, deformations and minimal obstruction theories from [Hal12b]. In Section 6, we give conditions on Aut, Def and Obs that imply the corresponding conditions on Exal needed in Theorem 4.4. In Section 7, we introduce n-step obstruction theories and conditions on them that can be used instead of the conditions on the minimal obstruction theory Obs. Finally, in Section 8 we prove the Main Theorem.

Notation. We follow standard conventions and notation. In particular, we adhere to the notation of [Hal12b]. Recall that if T is a scheme, then a point $t \in |T|$ is of *finite type* if $\operatorname{Spec}(\kappa(t)) \to T$ is of finite type. Points of finite type are locally closed. If $f : X \to Y$ is of finite type and $x \in |X|$ is of finite type, then $f(x) \in |Y|$ is of finite type.

1. Homogeneity, limit preservation, and extensions

In this section, we review the concept of homogeneity—a generalization of Schlessinger's Conditions that we attribute to J. Wise [Wis11, §2]—in the formalism of [Hal12b, §§1–2]. We will also briefly discuss limit preservation and extensions.

Fix a scheme S. An *S-groupoid* is a category X, together with a functor $a_X : X \to \mathbf{Sch}/S$ which is fibered in groupoids. A 1-morphism of S-groupoids $\Phi : (Y, a_Y) \to (Z, a_Z)$ is a functor between categories Y and Z that commutes strictly over $\mathbf{Sch}/S$. We will typically refer to an S-groupoid (X, a_X) as "X".

An *X-scheme* is a pair (T, σ_T), where T is an S-scheme and $\sigma_T : \mathbf{Sch}/T \to X$ is a 1-morphism of S-groupoids. A morphism of X-schemes $U \to V$ is a morphism of S-schemes $f : U \to V$ (which canonically determines a 1-morphism of S-groupoids $\mathbf{Sch}/f : \mathbf{Sch}/U \to \mathbf{Sch}/V$) together with a 2-morphism $\alpha : \sigma_U \Rightarrow \sigma_V \circ \mathbf{Sch}/f$. The collection of all X-schemes forms a 1-category, which we denote as $\mathbf{Sch}/X$. It is readily seen that $\mathbf{Sch}/X$ is an S-groupoid and that there is a natural equivalence of S-groupoids $\mathbf{Sch}/X \to X$. For a 1-morphism of S-groupoids $\Phi : Y \to Z$ there is an induced functor $\mathbf{Sch}/\Phi : \mathbf{Sch}/Y \to \mathbf{Sch}/Z$.

We will be interested in the following classes of morphisms of S-schemes:

- $\mathbf{Nil}$ – locally nilpotent closed immersions,
- $\mathbf{Cl}$ – closed immersions,
- $\mathbf{rNil}$ – morphisms $X \to Y$ such that there exists $(X_0 \to X) \in \mathbf{Nil}$ with the composition $(X_0 \to X \to Y) \in \mathbf{Nil}$,
- $\mathbf{rCl}$ – morphisms $X \to Y$ such that there exists $(X_0 \to X) \in \mathbf{Nil}$ with the composition $(X_0 \to X \to Y) \in \mathbf{Cl}$,
- $\mathbf{Art}^{\mathbf{fin}}$ – morphisms between local artinian schemes of finite type over S,
- $\mathbf{Art}^{\mathbf{insep}}$ – $\mathbf{Art}^{\mathbf{fin}}$-morphisms with purely inseparable residue field extensions,
- $\mathbf{Art}^{\mathbf{triv}}$ – $\mathbf{Art}^{\mathbf{fin}}$-morphisms with trivial residue field extensions,
- $\mathbf{Int}$ – integral morphisms,
- $\mathbf{Aff}$ – affine morphisms.

We certainly have a containment of classes of morphisms of S-schemes:

$$\begin{array}{ccccccc}
\mathbf{Nil} & \subset & \mathbf{Cl} & & & & \\
\cap & & \cap & & & & \\
\mathbf{rNil} & \subset & \mathbf{rCl} & \subset & \mathbf{Int} & \subset & \mathbf{Aff}. \\
\cup & & & & \cup & & \\
\mathbf{Art}^{\mathbf{triv}} & \subset & \mathbf{Art}^{\mathbf{insep}} & \subset & \mathbf{Art}^{\mathbf{fin}} & &
\end{array}$$

Fix a class P of morphisms that is contained in $\mathbf{Aff}$. In [Hal12b, §1], the notion of a P-homogeneous 1-morphism of S-groupoids $\Phi : Y \to Z$ was defined—an S-groupoid is P-homogeneous if its structure 1-morphism is. We will not recall the definition in full (it is somewhat lengthy), but we will give an explicit description in Lemma 1.1 for S-groupoids that are stacks in the Zariski topology.

An S-groupoid X, which is a Zariski stack, is *limit preserving*, if for any cofiltered system of affine S-schemes $\{\operatorname{Spec} A_i\}_{i\in I}$ with limit $\operatorname{Spec} A$, the natural functor:

$$\varinjlim_i X(\operatorname{Spec} A_i) \to X(\operatorname{Spec} A)$$

is an equivalence of categories [Art74, §1]. The definition just given also agrees with the definitions in [Hal12b, §4].

Lemma 1.1. *Fix a scheme S. Consider a class of morphisms $P \subset \mathbf{Aff}$ which is local for the Zariski topology. Let X be an S-groupoid, which is a stack for the Zariski topology. Then, the following conditions are equivalent:*

1. *X is P-homogeneous.*
2. *For any diagram of affine schemes $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \xrightarrow{\imath} \operatorname{Spec} A']$, where $\imath$ is a nilpotent closed immersion, $\operatorname{Spec} A \to \operatorname{Spec} B$ is P, then the natural functor:*
$$X(\operatorname{Spec}(A' \times_A B)) \to X(\operatorname{Spec} A') \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} B)$$
is an equivalence of categories.

If, in addition, X is limit preserving, and $P \in \{\mathbf{Nil}, \mathbf{Cl}, \mathbf{rNil}, \mathbf{rCl}, \mathbf{Int}, \mathbf{Aff}\}$, then in (2) it suffices to take $\operatorname{Spec} A$, $\operatorname{Spec} A'$, and $\operatorname{Spec} B$ to be locally of finite presentation over S. In particular, if S is locally noetherian, then $\mathbf{rCl}$-homogeneity is equivalent to the condition (S1$'$) of [Art74, 2.3].

Proof. The first part follows from the definitions. To see the second part, assume that X is limit preserving and that $P \in \{\mathbf{Nil}, \mathbf{Cl}, \mathbf{rNil}, \mathbf{rCl}, \mathbf{Int}, \mathbf{Aff}\}$. As X is a Zariski stack we may assume that $S = \operatorname{Spec}(R)$ is affine. Let $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \xrightarrow{\imath} \operatorname{Spec} A']$ be a diagram as in (2). Let $B' = B \times_A A'$. Then $B' \to B$ is surjective with nilpotent kernel so that $\operatorname{Spec} A' \to \operatorname{Spec} B'$ is a P-morphism. Now, by Proposition A.1 we may approximate $\operatorname{Spec} A' \to \operatorname{Spec} B'$ (resp. $\operatorname{Spec} B \to \operatorname{Spec} B'$) by P-morphisms (resp. nilpotent closed immersions) of finite presentation. A standard argument then shows that we may assume that $\operatorname{Spec} A' \to \operatorname{Spec} B'$ and $\operatorname{Spec} B \to \operatorname{Spec} B'$ are finitely presented.

Finally, we approximate B' by R-algebras B'_λ of finite type so that, for sufficiently large λ, we may descend $\operatorname{Spec} B \to \operatorname{Spec} B'$ and $\operatorname{Spec} A' \to \operatorname{Spec} B'$ to $\operatorname{Spec} B'_\lambda$ using [EGA, **IV**.8.10.5]. The result then follows from a simple argument. ∎

By [Wis11, Prop. 2.1], any algebraic stack is $\mathbf{Aff}$-homogeneous. It is easily verified, as is done in [*loc. cit.*], that if the stack is not necessarily algebraic but has representable diagonal, then the functor above is at least fully faithful. Moreover, $\mathbf{rCl}$-homogeneity is equivalent to Artin's *semi*-homogeneity condition [Art74, 2.2(S1a)] for X, its diagonal Δ_X, and its double diagonal Δ_{Δ_X}.

The main computational tool that P-homogeneity brings is [Hal12b, Lem. 1.4], which we now recall.

Lemma 1.2. *Fix a scheme S, a class of morphisms $P \subset \mathbf{Aff}$, a P-homogeneous S-groupoid X, and a diagram of X-schemes $[V \xleftarrow{p} T \xrightarrow{\imath} T']$, where $\imath$ is a locally* *nilpotent closed immersion and p is P. Then, there exists a cocartesian diagram in the category of X-schemes:*

$$\begin{tikzcd} T \arrow[r, hook, "\imath"] \arrow[d, "p"'] & T' \arrow[d, "p'"] \\ V \arrow[r, hook, "\imath'"] & V'. \end{tikzcd}$$

This diagram is also cocartesian in the category of S-schemes, the morphism $\imath'$ is a locally nilpotent closed immersion, p' is affine, and the induced homomorphism of sheaves:

$$\mathscr{O}_{V'} \to \imath'_*\mathscr{O}_V \times_{p'_*\imath_*\mathscr{O}_T} p'_*\mathscr{O}_{T'}$$

is an isomorphism.

Homogeneity supplies an S-groupoid with a quantity of linear data, which we now recall from [Hal12b, §2]. An X-*extension* is a square zero closed immersion of X-schemes $\imath : T \hookrightarrow T'$. The collection of X-extensions forms a category, which we denote as $\mathbf{Exal}_X$. There is a natural functor $\mathbf{Exal}_X \to \mathbf{Sch}/X : (\imath : T \hookrightarrow T') \to T$.

We denote by $\mathbf{Exal}_X(T)$ the fiber of the category $\mathbf{Exal}_X$ over the X-scheme T—we call these the X-extensions of T. There is a natural functor

$$\mathbf{Exal}_X(T)^\circ \to \mathbf{QCoh}(T) : (\imath : T \hookrightarrow T') \mapsto \ker(\imath^{-1}\mathscr{O}_{T'} \to \mathscr{O}_T).$$

We denote by $\mathbf{Exal}_X(T, I)$ the fiber category of $\mathbf{Exal}_X(T)$ over the quasicoherent $\mathscr{O}_T$-module I—we refer to these as the X-extensions of T by I.

Fix a scheme W and a quasicoherent $\mathscr{O}_W$-module J. We let $W[J]$ denote the W-scheme $\underline{\mathrm{Spec}}_W(\mathscr{O}_W[J])$ with structure morphism $r_{W,J} : W[J] \to W$. If W is an X-scheme, we consider $W[J]$ as an X-scheme via $r_{W,J}$. The X-extension $W \hookrightarrow W[J]$ is thus *trivial* in the sense that it allows an X-retraction.

By [Hal12b, Prop. 2.3], if the S-groupoid X is $\mathbf{Nil}$-homogeneous, then the groupoid $\mathbf{Exal}_X(T, I)$ is a Picard category. Denote the set of isomorphism classes of the category $\mathbf{Exal}_X(T, I)$ by $\mathrm{Exal}_X(T, I)$. Thus, we have additive functors

$$\mathrm{Der}_X(T, -) : \mathbf{QCoh}(T) \to \mathbf{Ab} : I \mapsto \mathrm{Aut}_{\mathbf{Exal}_X(T,I)}(T[I])$$
$$\mathrm{Exal}_X(T, -) : \mathbf{QCoh}(T) \to \mathbf{Ab} : I \mapsto \mathrm{Exal}_X(T, I).$$

We now record here the following easy consequences of [Hal12b, 2.2–2.5 & 3.5].

Lemma 1.3. *Fix a scheme S, an S-groupoid X, and an X-scheme T.*

1. *For a quasicoherent $\mathscr{O}_T$-module I, $\mathrm{Exal}_X(T, I) = 0$ if and only if for every X-extension $\imath : T \hookrightarrow T'$ of T by I, there is an X-morphism $r : T' \to T$ such that $r\imath = \mathrm{Id}_T$.*
2. *If X is $\mathbf{rNil}$-homogeneous, then the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is half-exact.*
3. *Suppose that X is a Zariski stack, $\mathbf{Nil}$-homogeneous, and limit preserving. If S is locally noetherian and T is locally of finite type over S, then the functor $M \mapsto \mathrm{Exal}_X(T, M)$ preserves filtered colimits.*
4. *For any* affine *étale morphism $p : U \to T$ and quasicoherent $\mathscr{O}_U$-module N, there is a natural functor $\psi : \mathbf{Exal}_X(T, p_*N) \to \mathbf{Exal}_X(U, N)$. If $(\imath : X \hookrightarrow X') \in \mathbf{Exal}_X(T, p_*N)$ with image $(\jmath : U \hookrightarrow U') \in \mathbf{Exal}_X(U, N)$,*

then there is a cartesian diagram of X-schemes

$$\begin{tikzcd} U \arrow[r, hook, "\jmath"] \arrow[d, "p"'] & U' \arrow[d, "p'"] \\ T \arrow[r, hook, "\imath"] & T', \end{tikzcd}$$

which is cocartesian as a diagram of S-schemes. If X is $\mathbf{Aff}$*-homogeneous, then ψ is an equivalence.*

Finally, we give conditions that imply $\mathbf{Art}^{\mathbf{fin}}$-homogeneity.

Lemma 1.4. *Fix a scheme S and an S-groupoid X that is* $\mathbf{Art}^{\mathbf{triv}}$*-homogeneous. Assume that one of the following conditions are satisfied.*

1. *X is a stack in the fppf topology.*
2. *X is a stack in the étale topology and* $\mathbf{Art}^{\mathbf{insep}}$*-homogeneous.*
3. *S is a $\mathbb{Q}$-scheme and X is a stack in the étale topology.*

Then X is $\mathbf{Art}^{\mathbf{fin}}$*-homogeneous.*

Proof. We start with noting that (3) implies (2). Now, let $[\operatorname{Spec} B \leftarrow \operatorname{Spec} A \hookrightarrow \operatorname{Spec} A']$ be a diagram of local artinian S-schemes, with $A' \twoheadrightarrow A$ a surjection of rings with nilpotent kernel, and $\operatorname{Spec} A \to \operatorname{Spec} B$ finite so that $\operatorname{Spec} A \to \operatorname{Spec} B$ belongs to $\mathbf{Art}^{\mathbf{fin}}$. Let $\operatorname{Spec} B' = \operatorname{Spec}(A' \times_A B)$ be the pushout of this diagram in the category of S-schemes. We have to prove that the functor

$$\varphi : X(\operatorname{Spec}(A' \times_A B)) \to X(\operatorname{Spec} A') \times_{X(\operatorname{Spec} A)} X(\operatorname{Spec} B)$$

is an equivalence. Assume that X is $\mathbf{Art}^{\mathbf{triv}}$-homogeneous (resp. $\mathbf{Art}^{\mathbf{insep}}$-homogeneous). We first show that φ is an equivalence when A, A' and B are not necessarily local but the residue field extensions of $\operatorname{Spec}(A) \to \operatorname{Spec}(B)$ are trivial (resp. purely inseparable). As $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B'$ is bijective, and X is a Zariski stack, we can work locally on $\operatorname{Spec} B'$ and assume that $\operatorname{Spec} B'$ is local. Then $\operatorname{Spec} B$ is also local and if we let $A = \prod_{i=1}^n A_i$ and $A' = \prod_{i=1}^n A'_i$ be decompositions such that $A' \twoheadrightarrow A_i$ factors through A'_i, then $\operatorname{Spec} B' = (A'_1 \times_{A_1} B) \times_B (A'_2 \times_{A_2} B) \times_B \cdots \times_B (A'_n \times_{A_n} B)$ is an iterated fiber product of local artinian rings. Equivalence of φ in the non-local case thus follows from the local case.

If X is a stack in the fppf (resp. étale) topology, then equivalence of φ is a local question on the fppf (resp. étale) topology on B' since fiber products of rings commute with flat base change. As $\operatorname{Spec} B \hookrightarrow \operatorname{Spec} B'$ is a nilpotent closed immersion, the scheme $\operatorname{Spec} B'$ is local artinian and the residue fields of B and B' coincide. Choose a finite (resp. finite separable) field extension K/k_B such that the residue fields of $k_A \otimes_{k_B} K$ are trivial (resp. purely inseparable) extensions of K. There is then a local artinian ring $\widetilde{B'}$ and a finite flat (resp. finite étale) extension $B' \hookrightarrow \widetilde{B'}$ with $k_{\tilde{B}'} = K$. Let $\widetilde{A} = A \times_{B'} \widetilde{B'}$ etc. Then $\widetilde{A}$, $\widetilde{A'}$, $\widetilde{B}$ are artinian rings such that all residue fields equal K (resp. are purely inseparable extensions of K). Thus, equivalence of φ follows from the case treated above. $\square$

2. Formal versality and formal smoothness

In this section we address a subtle point about the relationship between formal versality and formal smoothness. To be precise, we desire sufficient conditions for a family, formally versal at all *closed* points, to be formally *smooth*. In the algebraicity criterion for functors [Art69b, Thm. 5.3], a precise statement in this form is not present, but is addressed in [*op. cit.*, Lem. 5.4]. In the algebraicity criterion for groupoids [Art74, Thm. 5.3], the relevant result is precisely stated in [*op. cit.*, Prop. 4.2]. We do not, however, understand the proof.

In the notation of [*loc. cit.*], to verify formal smoothness, the residue fields of A are not fixed. But the proof of [*loc. cit.*] relies on [*op. cit.*, Thm. 3.3], which requires that the residue field of A is equal to the residue field of R. If the residue field extension is separable, then it is possible to conclude using [*op. cit.*, Prop. 4.3], which uses étale localization of obstruction theories. We do not know how to complete the argument if the residue field extension is inseparable. The essential problem is the verification that formal versality is smooth-local.

We also wish to point out that, in [*loc. cit.*], the techniques of Artin approximation are used via [*op. cit.*, Prop. 3.3]. In this section, we demonstrate that excellence (or related) assumptions are irrelevant with our formulation.

We begin this section with recalling, and refining, some results of [Hal12b, §4].

Definition 2.1. Fix a scheme S, an S-groupoid X, and an X-scheme T. Consider the following lifting problem: given a square zero closed immersion of X-schemes $Z_0 \hookrightarrow Z$ fitting into a commutative diagram of X-schemes:

$$\begin{tikzcd} Z_0 \arrow[r, "g"] \arrow[d, hook] & T \arrow[d] \\ Z \arrow[r] \arrow[ur, dashed] & X. \end{tikzcd}$$

The X-scheme T is:

formally smooth – if the lifting problem can always be solved étale locally on Z;

formally smooth at $t \in |T|$ – if the lifting problem can always be solved whenever the X-scheme Z is local artinian, with closed point z, such that $g(z) = t$, and the field extension $\kappa(t) \subset \kappa(z)$ is finite;

formally étale at $t \in |T|$ – if the lifting problem can always be solved *uniquely* whenever the X-scheme Z is local artinian, with closed point z, such that $g(z) = t$, and the field extension $\kappa(t) \subset \kappa(z)$ is finite;

formally versal at $t \in |T|$ – if the lifting problem can always be solved whenever the X-scheme Z is local artinian, with closed point z, such that $g(z) = t$, $\kappa(z) \cong \kappa(t)$, and there is an isomorphism of $\mathscr{O}_T$-modules $\kappa(t) \cong g_* \ker(\mathscr{O}_Z \to \mathscr{O}_{Z_0})$;

formally universal at $t \in |T|$ – if the lifting problem can always be solved *uniquely* whenever the X-scheme Z is local artinian, with closed point z, such that $g(z) = t$, $\kappa(z) \cong \kappa(t)$, and there is an isomorphism of $\mathscr{O}_T$-modules $\kappa(t) \cong g_* \ker(\mathscr{O}_Z \to \mathscr{O}_{Z_0})$.

We certainly have the following implications:

$$\begin{aligned} \text{formally smooth} &\Rightarrow \text{formally smooth at all } t \in |T| \\ &\Rightarrow \text{formally versal at all } t \in |T|. \end{aligned}$$

It is readily observed that formal smoothness is smooth local on the source. Without some more assumptions, it is not obvious to the authors that formal versality is smooth local on the source. Similarly, formal smoothness at t and formal versality at t are not obviously equivalent. We will see, however, that these subtleties vanish whenever the S-groupoid is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous. The notions of formally étale at a point, and formally universal will not be used in the sequel.

Lemma 2.2. *Fix a locally noetherian scheme S and a limit preserving S-groupoid X. Let T be an X-scheme which is locally of finite type over S and let $t \in |T|$ be a point such that:*

1. *T is formally smooth at $t \in |T|$;*
2. *the morphism $T \to X$ is representable by algebraic spaces.*

Then, for any X-scheme W, the morphism $T \times_X W \to W$ is smooth in a neighbourhood of every point over t. In particular, if T is formally smooth at every point of finite type*, then $T \to W$ is formally smooth.*

Proof. By a standard limit argument we can assume that $W \to S$ is of finite type. It is then enough to verify that $T \times_X W \to W$ is smooth at closed points above t and this follows from [EGA, **IV**.17.14.2]. The last statement follows from the fact that any closed point of $T \times_X W$ maps to a point of finite type of T. $\square$

There is a tight connection between formal smoothness (resp. formal versality) and X-extensions in the *affine* setting. Most of the next result was proved in [Hal12b, Lem. 4.3], which utilized arguments similar to those of [Fle81, Satz 3.2].

Lemma 2.3. *Fix a scheme S, an S-groupoid X, and an* affine *X-scheme T. Let $t \in |T|$ be a point. Consider the following conditions.*

1. *The X-scheme T is formally smooth at t.*
2. *The X-scheme T is formally versal at t.*
3. *$\mathrm{Exal}_X(T, \kappa(t)) = 0$.*

Then (1) $\Longrightarrow$ (2) and if X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous and t is of finite type, then (2) $\Longrightarrow$ (1). If X is $\mathbf{Cl}$-homogeneous, T is noetherian and t is a closed *point, then (2) $\Longrightarrow$ (3). If X is $\mathbf{rCl}$-homogeneous and t is a* closed *point, then (3) $\Longrightarrow$ (2).*

Remark 2.4. If X is $\mathbf{Aff}$-homogeneous and $\mathrm{Exal}_X(T, -) \equiv 0$, then T is formally smooth [Hal12b, Lem. 4.3] but we will not use this. If Exal_X commutes with Zariski-localization, that is, if for any open immersion of affine schemes $U \subseteq T$ the canonical map $\mathrm{Exal}_X(T, M) \otimes_{\Gamma(\mathscr{O}_T)} \Gamma(\mathscr{O}_U) \to \mathrm{Exal}_X(U, M|_U)$ is bijective, then the implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (2) also hold for non-closed points. This is essentially what Flenner proves in [Fle81, Satz 3.2] as his $\mathscr{E}x(T \to X, M)$ is the sheafification of the presheaf $U \mapsto \mathrm{Exal}_X(U, M|_U)$.

Proof of Lemma 2.3. The implication (1) $\Longrightarrow$ (2) follows from the definition and the implications (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (2) are proved in [Hal12b, Lem. 4.3]. The implication (2) $\Longrightarrow$ (1) follows from a similar argument: assume that T is formally versal at t and let $Z_0 \hookrightarrow Z$ be a square zero closed immersion of local artinian X-schemes fitting into a commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r] \arrow[d, hook] & T \arrow[d] \\ Z \arrow[r] & X, \end{tikzcd}$$

such that the closed point $z \in |Z_0|$ is mapped to $t \in |T|$ and $\kappa(z)/\kappa(t)$ is a finite extension. Let W_0 be the image of $Z_0 \to \mathrm{Spec}(\mathscr{O}_{T,t})$ so that W_0 is a local artinian scheme with residue field $\kappa(t)$. As X is $\mathbf{Art^{fin}}$-homogeneous, there is a commutative diagram

$$\begin{tikzcd} Z_0 \arrow[r] \arrow[d, hook] & W_0 \arrow[r] \arrow[d, hook] & T \arrow[d] \\ Z \arrow[r] & W \arrow[r] & X, \end{tikzcd}$$

where $W_0 \hookrightarrow W$ is a square zero closed immersion. As $W_0 \hookrightarrow W$ is a sequence of closed immersions with kernel isomorphic to $\kappa(t)$, there is a lift $W \to T$ and thus a lift $Z \to T$. $\square$

Thus, assuming that an S-groupoid X is $\mathbf{rCl}$-homogeneous, we can reformulate formal versality of an affine X-scheme T at a closed point $t \in |T|$ in terms of the triviality of the abelian group $\mathrm{Exal}_X(T, \kappa(t))$. Understanding the set of points $U \subset |T|$ where $\mathrm{Exal}_X(T, \kappa(u)) = 0$ for $u \in U$ will be accomplished in the next section. Combining the two lemmas above we obtain an analogue of [Art74, Prop. 4.2].

Proposition 2.5. *Fix a locally noetherian scheme S and a limit preserving, $\mathbf{Art^{fin}}$-homogeneous S-groupoid X. Let T be an X-scheme such that*
(1) *$T \to S$ is locally of finite type,*
(2) *$T \to X$ is formally versal at all points of finite type, and*
(3) *$T \to X$ is representable by algebraic spaces.*

Then $T \to X$ is formally smooth.

We also obtain the following result showing that formal versality is étale-local under mild hypotheses. This improves [Art74, Prop. 4.3] which requires the existence of an obstruction theory that is compatible with étale localization.

Proposition 2.6. *Fix a scheme S and an $\mathbf{Art^{triv}}$-homogeneous S-groupoid X that is a stack in the étale topology. Let $(U, u) \to (T, t)$ be an étale morphism of X-schemes. Then formal versality at (T, t) implies formal versality at (U, u).*

Proof. Reasoning as in the proof of Lemma 1.4, we see that X is homogeneous with respect to morphisms of artinian rings with separable residue field extensions. Arguing as in the proof of Lemma 2.3(2) $\Longrightarrow$ (1) we thus see that formal versality at (T, t) implies formal versality at (U, u). $\square$

Using Lemma 2.3, one can show that the converse holds if $u \in |U|$ and $t \in |T|$ are closed, X is $\mathbf{rCl}$-homogeneous, U and T are affine and noetherian, and $T \to X$ is representable by algebraic spaces.

Remark 2.7. Artin remarks [Art74, 4.9] that to verify the criteria for algebraicity it is enough to find suitable obstruction theories étale-locally. This assertion does not appear to be well grounded as his [Art74, Prop. 4.3] uses the existence of an obstruction theory. As our Proposition 2.6 does not use obstruction theories, on the other hand, it is enough to find obstruction theories étale-locally on T in the Main Theorem.

We end this section by giving conditions that ensure that if an X-scheme T is formally versal at all *closed* points, then it is formally versal at all points of *finite type*.

Condition 2.8 (Zariski localization of extensions)**.** For any open immersion of affine X-schemes $p : U \to T$, locally of finite type over S, and any point $u \in |U|$ of finite type, the map:
$$\mathrm{Exal}_X(T, \kappa(u)) \to \mathrm{Exal}_X(U, \kappa(u))$$
is surjective.

Lemma 1.3(4) implies that Condition 2.8 is satisfied whenever the S-groupoid X is **Aff**-homogeneous.

Proposition 2.9. *Fix a scheme S and an* **rCl***-homogeneous S-groupoid X satisfying Condition 2.8 (Zariski localization of extensions). Let T be an affine X-scheme, locally of finite type over S, and let $t \in |T|$ be a point of* finite type*. If* $\mathrm{Exal}_X(T, \kappa(t)) = 0$ *then the X-scheme T is formally versal at t.*

Proof. Finite type points are locally closed so there exists an open affine neighbourhood $U \subseteq T$ of t such that $t \in |U|$ is closed. By Condition 2.8 we have that $\mathrm{Exal}_X(U, \kappa(t)) = \mathrm{Exal}_X(T, \kappa(t)) = 0$ so the X-scheme U is formally versal at t by Lemma 2.3. It then follows, from the definition, that the X-scheme T also is formally versal at t. □

3. Vanishing and injectivity loci

Fix a scheme T. In this section, we will be interested in additive functors $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$. It is readily seen that the collection of all such functors forms an abelian category, with all limits and colimits computed "pointwise". For example, given additive functors $F, G : \mathbf{QCoh}(T) \to \mathbf{Ab}$ as well as a natural transformation $\varphi : F \to G$, then $\ker \varphi : \mathbf{QCoh}(T) \to \mathbf{Ab}$ is the functor
$$(\ker \varphi)(M) = \ker(F(M) \xrightarrow{\varphi(M)} G(M)).$$
Next, we set $A = \Gamma(\mathscr{O}_T)$. Note that the natural action of A on the abelian category $\mathbf{QCoh}(T)$ induces for every $M \in \mathbf{QCoh}(T)$ an action of A on the abelian group $F(M)$. Thus, we see that the functor F is canonically valued in the category $\mathbf{Mod}(A)$. It will be convenient to introduce the following notation: for a quasicompact and quasiseparated morphism of schemes $g : W \to T$ and a functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$, set $F_W : \mathbf{QCoh}(W) \to \mathbf{Ab}$ to be the functor $F_W(N) = F(g_*N)$. If F is additive (resp. preserves filtered colimits), then the same is true of F_W. The *vanishing locus of F* is the following subset ([Hal12a, §3]):
$$\begin{aligned}\mathbb{V}(F) &= \{t \in |T| \,:\, F(M) = 0 \quad \forall M \in \mathbf{QCoh}(T),\ \mathrm{Supp}(M) \subset \mathrm{Spec}(\mathscr{O}_{T,t})\} \\ &= \{t \in |T| \,:\, F_{\mathrm{Spec}(\mathscr{O}_{T,t})} \equiv 0\} \quad \text{(if T is quasi-separated).}\end{aligned}$$
The main result of this section, a criterion for the set $\mathbb{V}(F)$ to be Zariski open, is essentially due to H. Flenner [Fle81, Lem. 4.1]. In [*loc. cit.*], for an S-groupoid X and an affine X-scheme V, locally of finite type over S, a specific result about the vanishing locus of the functor $M \mapsto \mathrm{Exal}_X(V, M)$ is proved. In [*op. cit.*], the standing assumptions are that the S-groupoid X is *semi*-homogeneous, thus the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is only set-valued, which complicates matters. Since we are assuming **Nil**-homogeneity of X, the functor $M \mapsto \mathrm{Exal}_X(T, M)$ takes values in abelian groups. As we will see, this simplifies matters considerably.

Before we address vanishing loci of functors, the following simple application of Lazard's Theorem [Laz64], which appeared in [Hal12a, Prop. 3.2], will be a convenient tool to have at our disposal.

Proposition 3.1. *Fix an* affine *scheme $T = \mathrm{Spec}(A)$ and an additive functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$ that commutes with filtered colimits. Then, for any A-module M and any* flat *A-module L, the natural map:*

$$F(M) \otimes_A L \to F(M \otimes_A L)$$

is an isomorphism. In particular, for any A-algebra B and any flat B-module L, the natural map:

$$F(B) \otimes_B L \to F(L)$$

is an isomorphism.

We now make the following trivial observation.

Lemma 3.2. *Fix a scheme T and an additive functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$. Then the subset $\mathbb{V}(F) \subset |T|$ is stable under generization.*

By Lemma 3.2, we thus see that the subset $\mathbb{V}(F) \subset |T|$ will be Zariski open if we can determine sufficient conditions on the functor F and the scheme T so that the subset $\mathbb{V}(F)$ is (ind)constructible. We make the following definitions.

Definition 3.3. Fix an affine scheme T and an additive functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$.

- The functor F is *bounded* if the scheme T is noetherian and for any coherent $\mathscr{O}_T$-module M, the $\Gamma(\mathscr{O}_T)$-module $F(M)$ is coherent.
- The functor F is *weakly bounded* if the scheme T is noetherian and for any integral closed subscheme $\imath : T_0 \hookrightarrow T$, the $\Gamma(\mathscr{O}_{T_0})$-module $F(\imath_*\mathscr{O}_{T_0})$ is coherent.
- The functor F is *GI* (resp. *GS*, resp. *GB*) if there exists a dense open subset $U \subset |T|$ such that for all $u \in U$, the map
$$F(\mathscr{O}_T) \otimes_{\Gamma(\mathscr{O}_T)} \kappa(u) \to F(\kappa(u))$$
is injective (resp. surjective, resp. bijective).
- The functor F is *CI* (resp. *CS*, resp. *CB*) if for any integral closed subscheme $T_0 \hookrightarrow T$, the functor F_{T_0} is GI (resp. GS, resp. GB).

Functors of the above type occur frequently in algebraic geometry.

Example 3.4. Fix an affine noetherian scheme T and a complex $Q \in \mathsf{D}^-_{\mathrm{Coh}}(T)$. Then, for all $i \in \mathbb{Z}$, the functors on quasicoherent $\mathscr{O}_T$-modules given by $M \mapsto \mathrm{Ext}^i_{\mathscr{O}_T}(Q, M)$ and $M \mapsto \mathrm{Tor}_i^{\mathscr{O}_T}(Q, M)$ are additive, bounded, half-exact, commute with filtered colimits, and CB.

Example 3.5. Fix an affine noetherian scheme T and a morphism $p : X \to T$ which is projective and flat. Then, the functor $M \mapsto \Gamma(X, p^*M)$ is CB. Indeed, one version of the Cohomology and Base Change Theorem can be interpreted as saying that the functor $M \mapsto \Gamma(X, p^*M)$ is of the form given in Example 3.4.

We record for future reference a useful lemma.

Lemma 3.6. *Fix an affine noetherian scheme T and an additive functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$.*

(1) *If the functor F is* half-exact, *then F is bounded if and only if F is weakly bounded.*
(2) *If the functor F is (weakly) bounded, then any additive sub-quotient functor of F is (weakly) bounded.*
(3) *If F is GS (resp. CS), then so is any additive quotient functor of F.*
(4) *If F is weakly bounded and CI, then so is any additive subfunctor of F.*
(5) *Fix an exact sequence of additive functors $\mathbf{QCoh}(T) \to \mathbf{Ab}$:*

$$H_1 \longrightarrow H_2 \longrightarrow H_3 \longrightarrow H_4.$$

  (a) *If H_1 and H_3 are CS and H_4 is CI and weakly bounded, then H_2 is CS.*
  (b) *If H_1 is CS, H_2 and H_4 are CI, and H_4 is weakly bounded, then H_3 is CI.*

If the scheme T is noetherian and reduced, then (4), (5a)*, and* (5b) *hold with GI and GS instead of CI and CS.*

Proof. For claim (1), note that any coherent $\mathscr{O}_T$-module M admits a finite filtration whose successive quotients are of the form $\imath_*\mathscr{O}_{T_0}$, where $\imath : T_0 \hookrightarrow T$ is a closed immersion with T_0 integral. Induction on the length of the filtration, combined with the half-exactness of the functor F, proves the claim. Claims (2) and (3) are trivial. For (4), it is sufficient to prove the claim about GI and we can assume that T is a disjoint union of integral schemes. Fix an additive subfunctor $K \subset F$, then there is an exact sequence of additive functors: $0 \to K \to F \to H \to 0$. By (2) we see that H is weakly bounded and so $H(\mathscr{O}_T)$ is $\Gamma(\mathscr{O}_T)$-coherent. The ring $\Gamma(\mathscr{O}_T)$ is also reduced, so generic flatness implies that there is a dense open subset $U \subset |T|$ such that $\forall u \in U$, the $\Gamma(\mathscr{O}_T)$-module $H(\mathscr{O}_T)_u$ is flat. Thus, $\forall u \in U$ the sequence:

$$0 \longrightarrow K(\mathscr{O}_T) \otimes_{\Gamma(\mathscr{O}_T)} \kappa(u) \longrightarrow F(\mathscr{O}_T) \otimes_{\Gamma(\mathscr{O}_T)} \kappa(u) \longrightarrow H(\mathscr{O}_T) \otimes_{\Gamma(\mathscr{O}_T)} \kappa(u) \longrightarrow 0$$

is exact. By shrinking U, we may further assume that the map $F(\mathscr{O}_T)\otimes_{\Gamma(\mathscr{O}_T)}\kappa(u) \to F(\kappa(u))$ is injective. Hence, for all $u \in U$, from the commutative diagram:

$$\begin{tikzcd} K(\mathscr{O}_T) \otimes_{\Gamma(\mathscr{O}_T)} \kappa(u) \arrow[r, hook] \arrow[d] & F(\mathscr{O}_T) \otimes_{\Gamma(\mathscr{O}_T)} \kappa(u) \arrow[d, hook] \\ K(\kappa(u)) \arrow[r, hook] & F(\kappa(u)) \end{tikzcd}$$

we deduce that K is GI. Claims (5a) and (5b) follow from a similar argument to (4), combined with the 4-Lemmas of homological algebra. □

There is a remarkable Nakayama Lemma for half-exact functors, due to A. Ogus and G. Bergman [OB72, Thm. 2.1]. We state the following amplification, which follows from the mild strengthening given in [Hal12a, Cor. 3.5] and Lemma 3.2.

Theorem 3.7. *Fix an affine noetherian scheme T, and a half-exact, additive, and bounded functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$ which commutes with filtered colimits. Then,*

$$\mathbb{V}(F) = \{t \in |T| \,:\, F(\kappa(t)) = 0\}.$$

In particular, if $\mathbb{V}(F)$ contains all closed points, then $F \equiv 0$.

From this, we obtain a criterion for the openness of the vanishing locus, which is essentially [Fle81, Lem. 4.1].

Corollary 3.8. *Fix an affine noetherian scheme T and a half-exact, additive, and bounded functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$ which commutes with filtered colimits. If the functor F is CS, then the subset $\mathbb{V}(F) \subset |T|$ is Zariski open.*

Proof. By [EGA, **IV**.1.10.1], the set $\mathbb{V}(F)$ is open if and only if it is closed under generization and its intersection with an irreducible closed subset $T_0 \subset |T|$ contains a non-empty open subset or is empty. By Lemma 3.2, we have witnessed the stability under generization. Thus it remains to address the latter claim.

Fix an integral closed subscheme $T_0 \hookrightarrow T$. If $|T_0| \cap \mathbb{V}(F) \neq \emptyset$, then the generic point $\eta \in |T|$ of $|T_0|$ belongs to $\mathbb{V}(F)$ (Lemma 3.2). Thus, $F(\kappa(\eta)) = 0$. Since the functor F is, by assumption, CS, there exists a dense open subset $U_0 \subset |T_0|$ such that $\forall u \in U_0$ the map $F_{T_0}(\mathscr{O}_{T_0}) \otimes_{\Gamma(\mathscr{O}_{T_0})} \kappa(u) \to F(\kappa(u))$ is surjective.

As $\kappa(\eta)$ is a quasicoherent and flat $\mathscr{O}_{T_0}$-module, the natural map $F_{T_0}(\mathscr{O}_{T_0}) \otimes_{\Gamma(\mathscr{O}_{T_0})} \kappa(\eta) \to F(\kappa(\eta))$ is an isomorphism by Proposition 3.1. But $\eta \in \mathbb{V}(F)$, thus the coherent $\Gamma(\mathscr{O}_{T_0})$-module $F_{T_0}(\mathscr{O}_{T_0})$ is torsion. Hence, there is a dense open subset $U_0 \subset |T_0|$ with the property that if $u \in U_0$, then $F(\kappa(u)) = 0$. Using Theorem 3.7 we infer that $U_0 \subset \mathbb{V}(F) \cap |T_0|$. □

We conclude this section with a criterion for a functor to be GI (and consequently a criterion for a functor to be CI). This will be of use when we express Artin's criteria for algebraicity without obstruction theories.

Proposition 3.9. *Fix an affine and integral noetherian scheme T with function field K, and an additive functor $F : \mathbf{QCoh}(T) \to \mathbf{Ab}$ that commutes with filtered colimits such that $F(\mathscr{O}_T)$ is a coherent $\Gamma(\mathscr{O}_T)$-module. Then, the following are equivalent:*

1. *F is GI;*
2. *for any finite free quasicoherent $\mathscr{O}_T$-module M and $\eta \in F(M)$ such that for all non-zero maps $\epsilon : M \to K$ we have $\epsilon_*\eta \neq 0$ in $F(K)$, there exists a dense open subset $V_\eta \subset |T|$ such that for any non-zero map $\gamma : M \to \kappa(v)$, where $v \in V_\eta$, we have $\gamma_*\eta \neq 0$ in $F(\kappa(v))$.*

Proof. In this paper, we only need the implication (2)⇒(1), thus we omit the proof of the other direction. Set $R = \Gamma(\mathscr{O}_T)$ and $W = F(\mathscr{O}_T)$. We may shrink T sufficiently so that W becomes a finite free R-module. Then $W^\vee = \operatorname{Hom}_R(W, R)$ is also free. For any R-module Q there are natural homomorphisms

$$\sigma_Q : W \otimes_R Q \to \operatorname{Hom}_R(W^\vee, Q)$$
$$\delta_Q : W \otimes_R Q = F(\mathscr{O}_T) \otimes_R Q \to F(Q).$$

The first homomorphism is always an isomorphism and δ_Q is an isomorphism if Q is a flat R-module (Proposition 3.1). In particular, the maps δ_K and $\delta_{W^\vee}$ are isomorphisms.

Now, if $W = 0$, then the result is clear. Otherwise, we have that $W \otimes_R K = F(K) \neq 0$. Fix a map $\alpha : W^\vee \to Q$. Then the fundamental observation is that the following diagram commutes:

$$\begin{tikzcd} \mathrm{Hom}_R(W^\vee, W^\vee) \arrow[r, shift left, leftarrow, "\sigma_{W^\vee}"] \arrow[r, shift right, "\sigma_{W^\vee}^{-1}"'] \arrow[d, "\alpha_*"'] & W \otimes_R W^\vee \arrow[r, "\delta_{W^\vee}"] \arrow[d, "\mathrm{Id}_W \otimes \alpha"'] & F(W^\vee) \arrow[d, "\alpha_*"] \\ \mathrm{Hom}_R(W^\vee, Q) \arrow[r, shift left, leftarrow, "\sigma_Q"] \arrow[r, shift right, "\sigma_Q^{-1}"'] & W \otimes_R Q \arrow[r, "\delta_Q"'] & F(Q). \end{tikzcd}$$

Set $\xi = \delta_{W^\vee}\sigma_{W^\vee}^{-1}(\mathrm{Id}_{W^\vee}) \in F(W^\vee)$. Now consider a non-zero map $\epsilon : W^\vee \to K$. As δ_K is an isomorphism, the commutativity of the diagram says for $Q = K$:

$$\epsilon_*\xi = \epsilon_*\delta_{W^\vee}\sigma_{W^\vee}^{-1}(\mathrm{Id}_{W^\vee}) = \delta_K\sigma_K^{-1}(\epsilon) \neq 0 \in F(K).$$

By hypothesis, there exists a dense open $U \subset |T|$ such that for any non-zero map $\gamma : W^\vee \to \kappa(u)$ where $u \in U$ we have that $\gamma_*\xi \neq 0$. Now let $u \in U$, then for $a \in W \otimes_R \kappa(u)$ which is $\neq 0$ we have that the R-module homomorphism $\sigma_{\kappa(u)}(a) : W^\vee \to \kappa(u)$ is non-zero and $(\mathrm{Id}_W \otimes \sigma_{\kappa(u)}(a))(\xi) = a$. Also by hypothesis, $\sigma_{\kappa(u)}(a)_*(\xi) \neq 0 \in F(\kappa(u))$ and so the commutativity of the diagram says that $\delta_{\kappa(u)}(a) \neq 0 \in F(\kappa(u))$. Hence, the R-module homomorphism $W \otimes_R \kappa(u) \to F(\kappa(u))$ is injective for all $u \in U$ and F is GI. □

4. Openness of formal versality

As the title suggests, we now address the openness of the formally versal locus. Fix a scheme S. We now isolate the following conditions for a **Nil**-homogeneous S-groupoid X.

Condition 4.1 (Boundedness of extensions)**.** For any affine X-scheme T, locally of finite type over S, the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is bounded.

Condition 4.2 (Constructibility of extensions)**.** For any affine X-scheme T, locally of finite type over S, the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is CS.

To see that these conditions are plausible, observe the following

Lemma 4.3. *Fix a locally noetherian scheme S, an algebraic S-stack X, locally of finite type over S, and an affine X-scheme T, locally of finite type over S. Then, the functors $M \mapsto \mathrm{Der}_X(T, M)$ and $M \mapsto \mathrm{Exal}_X(T, M)$ are bounded and CB.*

Proof. By [Ols06, Thm. 1.1], there is a complex $L_{T/X} \in \mathsf{D}^-_{\mathrm{Coh}}(T)$ such that for all quasicoherent $\mathscr{O}_T$-modules M, there are natural isomorphisms $\mathrm{Der}_X(T, M) \cong \mathrm{Ext}^0_{\mathscr{O}_T}(L_{T/X}, M)$ and $\mathrm{Exal}_X(T, M) \cong \mathrm{Ext}^1_{\mathscr{O}_T}(L_{T/X}, M)$. The result follows from a consideration of Example 3.4. □

In their current form, Conditions 4.1 and 4.2 are difficult to verify. In §6, this will be rectified. In any case, we can now prove

Theorem 4.4. *Fix a locally noetherian scheme S and a limit preserving,* **rCl**-*homogeneous S-groupoid X satisfying Conditions 4.1 and 4.2 (boundedness and constructibility of extensions). Assume that either*

(1) *S is Jacobson; or*
(2) *X satisfies Condition 2.8 (Zariski-localization of extensions).*

Let T be an affine X-scheme, locally of finite type over S, and let $t \in |T|$ be a closed point at which T is formally versal. Then T is formally versal at every point of finite type *in a Zariski open neighbourhood of t. In particular, if X is also $\mathbf{Art}^{\mathbf{fin}}$-homogeneous, then T is formally smooth in a Zariski open neighbourhood of t.*

Proof. Fix a closed point $t \in |T|$ at which T is formally versal. By Condition 4.1 and Lemma 1.3, the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is bounded, half-exact, and preserves filtered colimits. Condition 4.2 now implies that the functor $M \mapsto \mathrm{Exal}_X(T, M)$ satisfies the criteria of Corollary 3.8. Thus, $\mathbb{V}(\mathrm{Exal}_X(T, -)) \subset |T|$ is a Zariski open subset. By Lemma 2.3(2) $\Longrightarrow$ (3) and Theorem 3.7, we have that $t \in \mathbb{V}(\mathrm{Exal}_X(T, -))$. So, there exists an open neighbourhood $t \in U \subset |T|$ with $\mathrm{Exal}_X(T, \kappa(u)) = 0$ for all $u \in U$. By Lemma 2.3(3) $\Longrightarrow$ (2), every closed point $u \in U$ is formally versal.

If S is Jacobson, then so is T and every point of finite type is closed. If instead X satisfies Condition 2.8, it follows from Proposition 2.9 that every point $u \in U$ of finite type is formally versal. The last assertion follows from Lemma 2.2. □

5. Automorphisms, deformations, and obstructions

In this section, we introduce the necessary deformation-theoretic framework that makes it possible to verify the hypotheses for the previous results. To do this, we recall the formulation of deformations and obstructions given in [Hal12b, §6].

Fix a scheme S and a 1-morphism of S-groupoids $\Phi : Y \to Z$. Define the category $\mathbf{Def}_\Phi$ to have objects the triples (T, J, η), where T is a Y-scheme, J is a quasicoherent $\mathscr{O}_T$-module, and η is a Y-scheme structure on the trivial Z-extension of T by J. Graphically, it is the category of completions of the following diagram:

$$\begin{tikzcd} T \arrow[r] \arrow[d, hook] & Y \arrow[d, "\Phi"] \\ T[J] \arrow[r] \arrow[ur, dashed, "\eta"] & Z. \end{tikzcd}$$

There is a natural functor $\mathbf{Def}_\Phi \to \mathbf{Sch}/Y : (T, J, \eta) \mapsto T$ and we denote the fiber of this functor over the Y-scheme T by $\mathbf{Def}_\Phi(T)$. There is also a functor $\mathbf{Def}_\Phi(T)^\circ \to \mathbf{QCoh}(T) : (J, \eta) \mapsto J$. We denote the fiber of this functor over a quasicoherent $\mathscr{O}_T$-module J as $\mathbf{Def}_\Phi(T, J)$. Note that this category is naturally pointed by the trivial Y-extension of T by J. Also, if the 1-morphism Φ is fibered in setoids, then the category $\mathbf{Def}_\Phi(T, J)$ is discrete. By [*op. cit.*, Prop. 8.3], if Y and Z are $\mathbf{Nil}$-homogeneous, then the groupoid $\mathbf{Def}_\Phi(T, J)$ is a Picard category. Denote the set of isomorphism classes of $\mathbf{Def}_\Phi(B, J)$ by $\mathrm{Def}_\Phi(B, J)$. Thus, we obtain $\Gamma(T, \mathscr{O}_T)$-linear functors:

$$\mathrm{Def}_\Phi(T, -) : \mathbf{QCoh}(T) \to \mathbf{Ab} : J \mapsto \mathrm{Def}_\Phi(T, J)$$
$$\mathrm{Aut}_\Phi(T, -) : \mathbf{QCoh}(T) \to \mathbf{Ab} : J \mapsto \mathrm{Aut}_{\mathbf{Def}_\Phi(T,J)}(T[J]).$$

The Lemma that follows is an easy consequence of [Hal12b, Lem. 6.2].

Lemma 5.1. *Fix a scheme S, a 1-morphism of $\mathbf{Cl}$-homogeneous S-groupoids $\Phi : Y \to Z$, and a Y-scheme T. Then, for any closed immersion $\imath : W \hookrightarrow T$ and quasicoherent $\mathscr{O}_W$-module N, the natural maps:*

$$\mathrm{Aut}_\Phi(T, \imath_* N) \to \mathrm{Aut}_\Phi(W, N) \quad \textit{and} \quad \mathrm{Def}_\Phi(T, \imath_* N) \to \mathrm{Def}_\Phi(W, N),$$

are isomorphisms.

We recall the exact sequence of [*op. cit.*, Prop. 8.5], which is our fundamental computational tool.

Proposition 5.2. *Fix a scheme S, a 1-morphism of* **Nil***-homogeneous S-groupoids $\Phi : Y \to Z$, a Y-scheme T, and a quasicoherent $\mathscr{O}_T$-module J. Then, there is a natural 6-term exact sequence of abelian groups:*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}_\Phi(T,J) \arrow[r] & \mathrm{Der}_Y(T,J) \arrow[r] & \mathrm{Der}_Z(T,J) \arrow[dll, rounded corners] \\ & \mathrm{Def}_\Phi(T,J) \arrow[r] & \mathrm{Exal}_Y(T,J) \arrow[r] & \mathrm{Exal}_Z(T,J). \end{tikzcd}$$

We now define $\mathrm{Obs}_\Phi(T,J) = \mathrm{coker}\big(\mathrm{Exal}_Y(T,J) \to \mathrm{Exal}_Z(T,J)\big)$ so that we obtain an $\Gamma(T,\mathscr{O}_T)$-linear functor:

$$\mathrm{Obs}_\Phi(T,-) : \mathbf{QCoh}(T) \to \mathbf{Ab} : J \mapsto \mathrm{Obs}_\Phi(T,J).$$

This is the *minimal obstruction theory* of Φ in the sense of Section 7.

Recall that if Φ is **rCl**-homogeneous then $\mathrm{Aut}_\Phi(T,-)$ and $\mathrm{Def}_\Phi(T,-)$ are half-exact [Hal12b, Cor. 6.4]. There is no reason to expect that $\mathrm{Obs}_\Phi(T,-)$ is half-exact though.

We finally recall [Hal12b, Prop. 6.9], which will show that the conditions of the Main Theorem are stable under composition, as well as permit proofs of these results by bootstrapping the diagonal.

Proposition 5.3. *Fix a scheme S, 1-morphisms of* **Nil***-homogeneous S-groupoids $X \xrightarrow{\Psi} Y \xrightarrow{\Phi} Z$, an X-scheme T, and a quasicoherent $\mathscr{O}_T$-module I.*

(1) *There is a natural 9-term exact sequence of abelian groups:*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Aut}_\Psi(T,I) \arrow[r] & \mathrm{Aut}_{\Phi\circ\Psi}(T,I) \arrow[r] & \mathrm{Aut}_\Phi(T,I) \arrow[dll, rounded corners] & \\ & \mathrm{Def}_\Psi(T,I) \arrow[r] & \mathrm{Def}_{\Phi\circ\Psi}(T,I) \arrow[r] & \mathrm{Def}_\Phi(T,I) \arrow[dll, rounded corners] & \\ & \mathrm{Obs}_\Psi(T,I) \arrow[r] & \mathrm{Obs}_{\Phi\circ\Psi}(T,I) \arrow[r] & \mathrm{Obs}_\Phi(T,I) \arrow[r] & 0. \end{tikzcd}$$

(2) *There are natural isomorphisms of abelian groups:*

$$\mathrm{Aut}_\Psi(T,I) \to \mathrm{Def}_{\Delta_\Psi}(T,I) \quad \textit{and} \quad \mathrm{Def}_\Psi(T,I) \to \mathrm{Obs}_{\Delta_\Psi}(T,I).$$

In particular, we may realize the functor $J \mapsto \mathrm{Def}_\Psi(T,J)$ as a 1-step relative obstruction theory for the 1-morphism Δ_Ψ.

(3) *Fix an* **rNil***-homogeneous 1-morphism of S-groupoids $W \to Y$, an X_W-scheme U, and a quasicoherent $\mathscr{O}_U$-module J. Then there is a natural injection*

$$\mathrm{Obs}_{\Psi_W}(U,J) \subset \mathrm{Obs}_\Psi(U,J).$$

In particular, we may realize the functor $J \mapsto \mathrm{Obs}_\Psi(U,J)$ as a 1-step relative obstruction theory for the 1-morphism $\Psi_W : X_W \to W$.

6. Relative conditions

Fix a locally noetherian scheme S. In this section we introduce a number of conditions for a 1-morphism of **Nil**-homogeneous S-groupoids $\Phi : Y \to Z$. These are the relative versions of some of the conditions which appear in the Main Theorem. For any of the conditions given in this section, a **Nil**-homogeneous S-groupoid X is said to have that condition, if the structure 1-morphism $X \to \mathbf{Sch}/S$ has the condition. These conditions are provided in the relative version so that this paper can be more readily seen to subsume the results of [Sta06].

Condition 6.1 (Boundedness of automorphisms)**.** For any affine and integral Y-scheme T_0, locally of finite type over S, the $\Gamma(\mathscr{O}_{T_0})$-module $\mathrm{Aut}_\Phi(T_0, \mathscr{O}_{T_0})$ is coherent.

Condition 6.2 (Boundedness of deformations)**.** For any affine and integral Y-scheme T_0, locally of finite type over S, the $\Gamma(\mathscr{O}_{T_0})$-module $\mathrm{Def}_\Phi(T_0, \mathscr{O}_{T_0})$ is coherent.

Condition 6.3 (Boundedness of obstructions)**.** For any affine Y-scheme T, locally of finite type over S, and for any integral closed subscheme $\imath : T_0 \hookrightarrow T$ the $\Gamma(\mathscr{O}_{T_0})$-module $\mathrm{Obs}_\Phi(T, \imath_*\mathscr{O}_{T_0})$ is coherent.

We note that Condition 6.3 often is satisfied for trivial reasons. If, for example, the S-groupoid Z satisfies Condition 4.1, which is the case when Z is algebraic, then Φ satisfies Condition 6.3.

Lemma 6.4. *Fix a locally noetherian scheme S and an* **rCl***-homogeneous S-groupoid X satisfying Condition 6.2. Then, X satisfies Condition 4.1 (boundedness of extensions).*

Proof. Fix an affine X-scheme $T = \operatorname{Spec} R$, locally of finite type over S. By Lemma 1.3(2), the functor $M \mapsto \mathrm{Exal}_X(T, M)$ is half-exact. Thus, by Lemma 3.6(1), it is sufficient to prove that for any integral closed subscheme $\imath : T_0 \hookrightarrow T$, the R-module $\mathrm{Exal}_X(T, \imath_*\mathscr{O}_{T_0})$ is coherent. Now, by Proposition 5.2, there is an exact sequence:

$$\mathrm{Def}_{X/S}(T, \imath_*\mathscr{O}_{T_0}) \longrightarrow \mathrm{Exal}_X(T, \imath_*\mathscr{O}_{T_0}) \longrightarrow \mathrm{Exal}_S(T, \imath_*\mathscr{O}_{T_0}).$$

By Lemma 4.3, the R-module $\mathrm{Exal}_S(T, \imath_*\mathscr{O}_{T_0})$ is coherent, and by Lemma 5.1, we have that $\mathrm{Def}_{X/S}(T, \imath_*\mathscr{O}_{T_0}) = \mathrm{Def}_{X/S}(T_0, \mathscr{O}_{T_0})$, which is a coherent $\Gamma(\mathscr{O}_{T_0})$-module by Condition 6.2. It follows that $\mathrm{Exal}_X(T, \imath_*\mathscr{O}_{T_0})$ is a coherent R-module from the exact sequence. □

Similarly, to expand Condition 4.2 (constructibility of extensions), we introduce the following conditions.

Condition 6.5 (Constructibility of automorphisms)**.** For any affine, integral Y-scheme T_0, locally of finite type over S, the functor $\mathrm{Aut}_\Phi(T_0, -)$ is GB.

Condition 6.6 (Constructibility of deformations)**.** For any affine, integral Y-scheme T_0, locally of finite type over S, the functor $\mathrm{Def}_\Phi(T_0, -)$ is GB.

Condition 6.7 (Constructibility of obstructions)**.** For any affine Y-scheme T, locally of finite type over S, the functor $\mathrm{Obs}_\Phi(T, -)$ is CI.

Lemma 6.8. *Fix a locally noetherian scheme S and a 1-morphism of* **Cl**-*homogeneous S-groupoids $\Phi : Y \to Z$ satisfying Conditions 6.3, 6.6 and 6.7. If Z satisfy Condition 4.2 (constructibility of extensions), then so does Y.*

Proof. By Proposition 5.2, we have an exact sequence of additive functors $\mathbf{QCoh}(T) \to \mathbf{Ab}$:

$$\mathrm{Def}_\Phi(T,-) \longrightarrow \mathrm{Exal}_Y(T,-) \longrightarrow \mathrm{Exal}_Z(T,-) \longrightarrow \mathrm{Obs}_\Phi(T,-) \longrightarrow 0.$$

For any integral closed subscheme $\imath : T_0 \hookrightarrow T$, we have that $\mathrm{Def}_\Phi(T_0,-) = \mathrm{Def}_\Phi(T,\imath_*(-))$ by Lemma 5.1. Condition 6.6 gives that $\mathrm{Def}_\Phi(T_0,-)$ is GS so the functor $\mathrm{Def}_\Phi(T,-)$ is CS. The other conditions imply that $\mathrm{Exal}_Z(T,-)$ is CS and that $\mathrm{Obs}_\Phi(T,-)$ is CI and weakly bounded. Thus, Lemma 3.6(5a) shows that the functor $\mathrm{Exal}_Y(T,-)$ is CS. □

We now move on and address Condition 2.8 (Zariski localization of extensions).

Condition 6.9 (Zariski localization of automorphisms)**.** For any affine, integral Y-scheme T_0, locally of finite type over S, any affine open subscheme $U_0 \subset T_0$, and any point $u \in |U_0|$ of finite type, the natural map

$$\mathrm{Aut}_\Phi(T_0,\kappa(u)) \to \mathrm{Aut}_\Phi(U_0,\kappa(u)).$$

is bijective.

Condition 6.10 (Zariski localization of deformations)**.** For any affine, integral Y-scheme T_0, locally of finite type over S, any affine open subscheme $U_0 \subset T_0$, and any point $u \in |U_0|$ of finite type, the natural map

$$\mathrm{Def}_\Phi(T_0,\kappa(u)) \to \mathrm{Def}_\Phi(U_0,\kappa(u)).$$

is bijective.

Condition 6.11 (Zariski localization of obstructions)**.** For any affine Y-scheme T, locally of finite type over S, any affine open subscheme $U \subset T$, and any point $u \in |U|$ of finite type, the natural map

$$\mathrm{Obs}_\Phi(T,\kappa(u)) \to \mathrm{Obs}_\Phi(U,\kappa(u))$$

is injective.

The proof of the next result is similar, but easier, than the proof of Lemma 6.8, thus is omitted.

Lemma 6.12. *Fix a scheme S and a 1-morphism of* **Cl**-*homogeneous S-groupoids $\Phi : Y \to Z$ satisfying Conditions 6.10 and 6.11. If Z satisfies Condition 2.8 (Zariski localization of extensions), then so does Y.*

7. Obstruction theories

As in the previous section, we let S be a locally noetherian scheme and let $\Phi : Y \to Z$ be a 1-morphism of **Nil**-homogeneous S-groupoids. We will expand the conditions on obstructions and obtain conditions that are more readily verifiable. We begin with recalling the definition of an n-step relative obstruction theory given in [Hal12b, Defn. 6.6].

An *n-step relative obstruction theory for Φ*, denoted $\{\mathrm{o}^i(-,-),\mathrm{O}^i(-,-)\}_{i=1}^n$, is for each Y-scheme T, a sequence of additive functors (the obstruction spaces):

$$\mathrm{O}^i(T,-) : \mathbf{QCoh}(T) \to \mathbf{Ab} : J \mapsto \mathrm{O}^i(T,J) \quad i = 1,\dots,n$$

as well as natural transformations of functors (the obstruction maps):

$$\mathrm{o}^1(T,-) : \mathrm{Exal}_Z(T,-) \Rightarrow \mathrm{O}^1(T,-)$$
$$\mathrm{o}^i(T,-) : \ker \mathrm{o}^{i-1}(T,-) \Rightarrow \mathrm{O}^i(T,-) \quad \text{for } i = 2, \dots, n,$$

such that the natural transformation of functors:

$$\mathrm{Exal}_Y(T,-) \Rightarrow \mathrm{Exal}_Z(T,-)$$

has image $\ker \mathrm{o}^n(T,-)$. Furthermore, we say that the obstruction theory is

- *(weakly) bounded*, if for any affine Y-scheme T, locally of finite type over S, the obstruction spaces $M \mapsto \mathrm{O}^i(T,M)$ are (weakly) bounded functors;
- *Zariski- (resp. étale-) functorial* if for any open immersion (resp. étale morphism) of affine Y-schemes $g : V \to T$, and $i = 1, \dots, n$, there is a natural transformation of functors:

  $$C_g^i : \mathrm{O}^i(T, g_*(-)) \Rightarrow \mathrm{O}^i(V,-),$$

  which for any quasicoherent $\mathscr{O}_V$-modules N, make the following diagrams commute:

$$\begin{array}{ccc} \mathrm{Exal}_X(T,g_*N) & \longrightarrow & \mathrm{O}^1(T,g_*N) \\ \downarrow & & \downarrow \\ \mathrm{Exal}_X(V,N) & \longrightarrow & \mathrm{O}^1(V,N) \end{array} \qquad \begin{array}{ccc} \ker \mathrm{o}^{i-1}(T,g_*N) & \longrightarrow & \mathrm{O}^i(T,g_*N) \\ \downarrow & & \downarrow \\ \ker \mathrm{o}^{i-1}(V,N) & \longrightarrow & \mathrm{O}^i(V,N). \end{array}$$

  We also require for any open immersion (resp. étale morphism) of affine schemes $h : W \to V$, an isomorphism of functors:

  $$\alpha^i_{g,h} : C^i_h \circ C^i_g \Rightarrow C^i_{gh}.$$

Remark 7.1 (Comparison with Artin's obstruction theories). An obstruction theory in the sense of [Art74, 2.6] is a 1-step bounded obstruction theory "that is functorial in the obvious sense". We take this to mean étale-functorial in the above sense. Obstruction theories are usually half-exact and functorial for any morphism, but Exal is only contravariantly functorial for étale morphisms so the condition above does not make sense for arbitrary morphisms. On the other hand, for **Aff**-homogeneous stacks, Exal is *covariantly* functorial for any morphism, cf. [Hal12b, Proof of Cor. 2.5]. Also note that the minimal obstruction theory Obs_Φ is étale-functorial.

We have the following simple

Lemma 7.2. *Fix a locally noetherian scheme S and a 1-morphism of* **Nil***-homogeneous S-groupoids $\Phi : Y \to Z$. Let $\{\mathrm{o}^i, \mathrm{O}^i\}_{i=1}^n$ be an n-step relative obstruction theory for Φ. Let $\widetilde{\mathrm{O}}^i(T,M) \subset \mathrm{O}^i(T,M)$ be the image of $\mathrm{o}^i(T,M)$ for $i = 1, \dots, n$. Then $\{\mathrm{o}^i, \widetilde{\mathrm{O}}^i\}_{i=1}^n$ is an n-step relative obstruction theory for Φ. Moreover, let $\mathrm{Obs}^i(T,-) = \mathrm{Exal}_Z(T,-)/\ker \mathrm{o}^i$ and $\mathrm{Obs}^0(T,-) = 0$. Then $\mathrm{Obs}^n(T,-) = \mathrm{Obs}_\Phi(T,-)$ and we have exact sequences*

$$0 \longrightarrow \widetilde{\mathrm{O}}^i(T,-) \longrightarrow \mathrm{Obs}^i(T,-) \longrightarrow \mathrm{Obs}^{i-1}(T,-) \longrightarrow 0.$$

for $i = 1, 2, \dots, n$. In particular, if the obstruction theory is (weakly) bounded, then so is the minimal obstruction theory $\mathrm{Obs}_\Phi(T,-)$.

We now introduce the following two variants of Condition 6.7 (constructibility of obstructions). The first is with an n-step relative obstruction theory. The second is without reference to a linear obstruction theory, just as in [Art69b, Thm. 5.3 [5′c]] and [Sta06].

Condition 7.3 (Constructibility of obstructions II)**.** There exists a weakly bounded, n-step relative obstruction theory for Φ, $\{\mathrm{o}^i(-,-), \mathrm{O}^i(-,-)\}_{i=1}^n$, such that for each affine Y-scheme T, locally of finite type over S, and $i = 1$, …, n, the obstruction spaces $M \mapsto \mathrm{O}^i(T, M)$ are CI.

Condition 7.4 (Constructibility of obstructions III)**.** For any affine Y-scheme T, locally of finite type over S, any integral closed subscheme $T_0 \hookrightarrow T$ (with fraction field K_0) and any finite free $\mathscr{O}_{T_0}$-module M_0, then given a Z-extension $T \hookrightarrow T'$ of T by M_0 such that for any non-zero map $\epsilon : M_0 \to K_0$, the resulting Z-extension of T by K_0 is obstructed, there exists a dense open subset $U_0 \subset |T_0|$ such that for all $u \in U_0$ and all non-zero maps $\gamma : M_0 \to \kappa(u)$, the resulting Z-extension of T by $\kappa(u)$ is obstructed.

Lemma 7.5. *Fix a locally noetherian scheme S and a 1-morphism of* **Nil***-homogeneous S-groupoids $\Phi : Y \to Z$. If Φ satisfies Condition 6.3 (boundedness of obstructions), then Conditions 6.7, 7.3 and 7.4 are equivalent. Moreover, Condition 7.3 implies Condition 6.3.*

Proof. Condition 7.4 is an expansion of condition (2) in Proposition 3.9 for the functor $M \mapsto \mathrm{Obs}_\Phi(T, M)$. The equivalence of Conditions 6.7 and 7.4 thus follows from Proposition 3.9. If Condition 6.7 holds, then the minimal obstruction theory satisfies 7.3. It is thus enough to show that Condition 7.3 implies Condition 6.7.

If the functors $\mathrm{O}^i(T, -)$ are CI and weakly bounded then so are the subfunctors $\widetilde{\mathrm{O}}^i(T, -)$ of Lemma 7.2 by Lemma 3.6(4). Since $\mathrm{Obs}_\Phi(T, -)$ is an iterated extension of the $\widetilde{\mathrm{O}}^i(T, -)$'s, it follows that $\mathrm{Obs}_\Phi(T, -)$ is CI and weakly bounded by Lemma 3.6(5b) so that Conditions 6.7 and 6.3 hold. □

We continue with an expansion of Condition 6.11 (Zariski localization of obstructions). As before, the first is in terms of an n-step relative obstruction theory and the second is without reference to a linear obstruction theory as in [Sta06].

Condition 7.6 (Zariski localization of obstructions II)**.** There exists a functorial, n-step relative obstruction theory for Φ, $\{\mathrm{o}^i(-,-), \mathrm{O}^i(-,-)\}_{i=1}^n$, such that for any affine Y-scheme T, locally of finite type over S, and any open subscheme $U \subset T$ and point $u \in |U|$ of finite type, then for all $i = 1$, …, n, the canonical maps:

$$\mathrm{O}^i(T, \kappa(u)) \to \mathrm{O}^i(U, \kappa(u))$$

are injective.

Condition 7.7 (Zariski localization of obstructions III)**.** For any affine Y-scheme T, locally of finite type over S, and any finite type point $t \in |T|$, if a Z-extension T by $\kappa(t)$ does not lift to a Y-extension, then for any affine open neighbourhood $U \subset T$ of t, the induced Z-extension of U by $\kappa(t)$ does not lift to a Y-extension.

Lemma 7.8. *Fix a locally noetherian scheme S and a 1-morphism of* **Nil***-homogeneous S-groupoids $\Phi : Y \to Z$. Then Conditions 6.11, 7.6 and 7.7 for Φ are equivalent.*

Proof. Condition 7.7 is just an expansion of Condition 6.11 and if Condition 6.11 holds then the minimal obstruction theory satisfies 7.6. That Condition 7.6 implies Condition 6.11 follows from Lemma 7.2. □

8. Proof of Main Theorem

We now obtain the following algebraicity criterion for groupoids.

Proposition 8.1. *Fix an excellent scheme S. Then, an S-groupoid X is an algebraic S-stack, locally of finite presentation over S, if and only if it satisfies the following conditions.*

1. *X is a stack over $(\mathbf{Sch}/S)_{\text{Ét}}$.*
2. *X is limit preserving.*
3. *X is $\mathbf{Art}^{\mathbf{insep}}$- and $\mathbf{rCl}$-homogeneous.*
4. *X is effective, i.e., Condition (4) of Main Theorem.*
5. *The diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is representable.*
6. *Condition 6.2 (boundedness of deformations).*
7. *Condition 4.2 (constructibility of extensions).*
8. *Condition 2.8 (Zariski localization of extensions) or S is Jacobson.*

Proof. Just as in the proof of [Hal12b, Cor. 4.6], conditions (2)–(4)—using only $\mathbf{Art}^{\mathbf{triv}}$-homogeneity—together with Condition 6.2 permit the application of [CJ02, Thm. 1.5]. Thus, for any pair $(\operatorname{Spec} \Bbbk \xrightarrow{x} S, \xi)$, where $\Bbbk$ is a field, x is a morphism locally of finite type, and $\xi \in X(x)$, there exists a pointed and affine X-scheme (Q_ξ, q) such that Q_ξ is locally of finite type over S, there is an isomorphism of X-schemes $\operatorname{Spec} \kappa(q) \cong \operatorname{Spec} \Bbbk$, and Q_ξ is a formally versal X-scheme at the closed point q.

As X is $\mathbf{rCl}$-homogeneous and satisfies Condition 6.2, Lemma 6.4 implies that X satisfies Condition 4.1 (boundedness of extensions). By Lemma 1.4, X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous. Using Conditions 4.1 and 4.2 (boundedness and constructibility of extensions), condition (8) and $\mathbf{Art}^{\mathbf{fin}}$-homogeneity, it follows from Theorem 4.4 that we are free to assume—by passing to an affine open neighbourhood of q—that the X-scheme Q_ξ is formally smooth at every closed point.

The remainder of the proof of [Hal12b, Cor. 4.6] applies without change. □

Lemma 8.2. *Fix a locally noetherian scheme S and an S-groupoid X satisfying the following conditions.*

1. *X is an étale stack.*
2. *X is limit preserving.*
3. *The diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is representable.*
4. *Fix a field $\Bbbk$, a morphism $x : \operatorname{Spec} \Bbbk \to S$ which is locally of finite type, and $\xi \in F(x)$. Then, there is an affine X-scheme T, locally of finite type over S, which is formally smooth at a closed point $t \in |T|$, such that the X-schemes ξ and $\operatorname{Spec} \kappa(t)$ are isomorphic.*

Then, F is $\mathbf{Int}$-homogeneous.

Proof. Since X is a Zariski stack that is also limit preserving, to show that X is $\mathbf{Int}$-homogeneous, it suffices, by Lemma 1.1, to prove that: for any diagram of affine S-schemes $[\operatorname{Spec} A_2 \leftarrow \operatorname{Spec} A_0 \hookrightarrow \operatorname{Spec} A_1]$, locally of finite type over S, where

$A_1 \twoheadrightarrow A_0$ is surjective with nilpotent kernel and $A_0 \to A_2$ is finite, the canonical map

$$X(\operatorname{Spec}(A_2 \times_{A_0} A_1)) \to X(\operatorname{Spec} A_2) \times_{X(\operatorname{Spec} A_0)} X(\operatorname{Spec} A_1)$$

is an equivalence. Set $A_3 = A_2 \times_{A_0} A_1$ and for $i = 0, \ldots, 3$ let $W_i = \operatorname{Spec} A_i$. Then, we must uniquely complete all commutative diagrams:

$$\begin{tikzcd}
& W_1 \arrow[dr] \arrow[drr, bend left] & & \\
W_0 \arrow[ur, hook, "\imath"] \arrow[dr, "p"'] & & W_3 \arrow[r, dashed] & X \\
& W_2 \arrow[ur, hook] \arrow[urr, bend right] & &
\end{tikzcd}$$

Since X is an étale stack, the problem of constructing a map $W_3 \to X$ is étale local on W_3. Thus, it is sufficient to construct for each point $w_3 \in |W_3|$ a morphism of pointed schemes $(U_3, u_3) \to (W_3, w_3)$, which is smooth, together with a unique map $U_3 \to X$ which is compatible with pulling back the square above by $U_3 \to W_3$.

The morphism $W_2 \to W_3$ is a nilpotent closed immersion, so w_3 is in the image of a unique point w_2 of W_2 and $\kappa(w_2) \cong \kappa(w_3)$. The points of W_2 which are of finite type over S are dense, thus the same is true of W_3. So, we may assume that the morphism $\operatorname{Spec}\kappa(w_3) \to S$ is locally of finite type.

By condition (4), there exists an affine X-scheme T, locally of finite type over S, which is formally smooth at a closed point $t \in |T|$, and the X-schemes $\operatorname{Spec}\kappa(t)$ and $\operatorname{Spec}\kappa(w_3)$ are isomorphic. Let $W_i' = W_i \times_X T$ for $i = 0, 1, 2$. By (3), the morphism $T \to F$ is representable so, by Lemma 2.2, the pull-back $W_i' \to W_i$ is smooth in a neighbourhood of the inverse image of t. Let $W_i'^{\mathrm{sm}} \subset W_i'$ be the smooth locus of $W_i' \to W_i$. We then let $W_2'' = W_2'^{\mathrm{sm}} \setminus p(W_0' \setminus \imath^{-1}(W_1'^{\mathrm{sm}}))$ and $W_0'' = p^{-1}(W_2'')$ and $W_1'' = \imath(W_0'')$ as open subsets of $W_i'^{\mathrm{sm}}$. Note that, since p is integral, hence closed, all points above t belong to the W_i''. In particular, we have that w_2 is in the image of W_2''.

By [Hal12b, Lem. A.4], there is a commutative diagram of S-schemes:

$$\begin{tikzcd}
& W_0'' \arrow[rr, hook] \arrow[dl] \arrow[dd] & & W_1'' \arrow[dl] \arrow[dd] \\
W_2'' \arrow[rr, hook] \arrow[dd] & & W_3'' \arrow[dd] & \\
& W_0 \arrow[rr, hook] \arrow[dl] & & W_1 \arrow[dl] \\
W_2 \arrow[rr, hook] & & W_3 &
\end{tikzcd}$$

where all faces of the cube are cartesian, the top and bottom faces are cocartesian, and the map $W_3'' \to W_3$ is flat. By [Hal12b, Lem. A.5], the morphism is $W_3'' \to W_3$ is smooth. Since the top square is cocartesian, and there are compatible maps $W_i'' \to T$ for $i \neq 3$, there is a uniquely induced map $W_3'' \to T$. Taking the composition of this map with $T \to X$, we obtain a map $W_3'' \to X$ which is compatible with the data. This map is unique because the diagonal of X is representable. As $W_3'' \to W_3$ is an étale neighbourhood of w_3 we are done. $\square$

We are now ready to prove the Main Theorem.

Proof of Main Theorem. Repeating the bootstrapping techniques of the proof of [Hal12b, Thm. A], it is sufficient to prove the result in the case where the diagonal 1-morphism $\Delta_{X/S} : X \to X \times_S X$ is representable.

As in the first part of the proof of Proposition 8.1, we see that for every point $x : \operatorname{Spec}(\Bbbk) \to X$, of finite type over S, there is an affine X-scheme (Q_ξ, q) such that Q_ξ is locally of finite type over S, there is an isomorphism of X-schemes $\operatorname{Spec} \kappa(q) \cong \operatorname{Spec} \Bbbk$, and Q_ξ is a formally versal X-scheme at the closed point q. Now since X is $\mathbf{Art}^{\mathbf{fin}}$-homogeneous (Lemma 1.4), it follows that Q_ξ is formally smooth at the closed point q by Lemma 2.3. Then, by Lemma 8.2, we see that the S-presheaf F is $\mathbf{Int}$-homogeneous and thus also $\mathbf{rCl}$-homogeneous. So, by Proposition 8.1, it remains to show that the hypotheses of the Theorem guarantee that Conditions 4.2 and 2.8 (constructibility and Zariski localization of extensions) hold for X.

Now, by Lemmata 4.3 and 1.3(4) we have that Conditions 4.1, 4.2 and 2.8 (boundedness, constructibility and Zariski localization of extensions) hold for S. Trivially, Condition 6.3 (boundedness of obstructions) then holds for X. Thus, by Lemmata 6.8 and 7.5, together with the hypotheses (5(b)ii) and (6a), we see that Condition 4.2 holds for X. Similarly, by the hypotheses (5(c)ii) and (6b), as well as Lemmata 6.12 and 7.8, we see that X satisfies Condition 2.8.

We may thus apply Proposition 8.1 to conclude that X is an algebraic stack, locally of finite presentation over S. $\square$

Appendix A. Approximation of integral morphisms

In this appendix we give a approximation result for integral homomorphisms. It is somewhat technical since the properties that we need, surjective and surjective with nilpotent kernel, cannot be deduced for an arbitrary approximation but the approximation has to be built with these properties in mind.

Proposition A.1. *Let A be a ring, let B be an A-algebra and let C be an B-algebra. Assume that B and C are integral A-algebras. Then there exists a filtered system $(B_\lambda \to C_\lambda)_\lambda$ of finite and finitely presented A-algebras, with colimit $B \to C$. In addition, if $A \to B$ (resp. $B \to C$, resp. $A \to C$) has one of the properties:*

1. *surjective,*
2. *surjective with nilpotent kernel,*

then $A \to B_\lambda$ (resp. $B_\lambda \to C_\lambda$, resp. $A \to C_\lambda$) has the corresponding property.

Proof. We begin by writing $B = \varinjlim_{\lambda \in \Lambda} B_\lambda^\circ$ and $C = \varinjlim_{\lambda\in\Lambda} C_\lambda^\circ$ as colimits of finitely generated subalgebras. We may then replace C_λ° with the C-subalgebra generated by the image of B_λ° and C_λ° so that we have homomorphisms $B_\lambda^\circ \to C_\lambda^\circ$ for all λ. If $B \to C$ is surjective, then we let C_λ° be the image of $B_\lambda^\circ \to C$. It is now easily verified that if $A \to B$ (resp. $B \to C$, resp. $A \to C$) is surjective or surjective with nilpotent kernel then so is $A \to B_\lambda^\circ$ (resp. $B_\lambda^\circ \to C_\lambda^\circ$, resp. $A \to C_\lambda^\circ$).

For every λ, choose surjections $P_\lambda \to B_\lambda^\circ$ and $Q_\lambda \to C_\lambda^\circ$ where P_λ and Q_λ are finite and finitely presented A-algebras. We may assume that we have homomorphisms $P_\lambda \to Q_\lambda$ compatible with $B_\lambda^\circ \to C_\lambda^\circ$ and if $B \to C$ is surjective, then we take $P_\lambda = Q_\lambda$. For any finite subset $L \subseteq \Lambda$ let $P_L = \bigotimes_{\lambda \in L} P_\lambda$ and $Q_L = \bigotimes_{\lambda\in L} Q_\lambda$, where the tensor products are over A.

For fixed $L \subseteq \Lambda$ choose finitely generated ideals $I_L \subseteq \ker(P_L \to B)$ and $I_L Q_L \subseteq J_L \subseteq \ker(Q_L \to C)$ and let $B_L = P_L/I_L$ and $C_L = Q_L/J_L$. If $A \to B$ (resp. $A \to C$) is surjective, then for sufficiently large I_L (resp. J_L), we have that $A \to B_L$ (resp. $A \to C_L$) is surjective. If $B \to C$ is surjective, then by construction $P_L = Q_L$

so that $B_L \to C_L$ is surjective. If $B \to C$ has nilpotent kernel, with nilpotency index n, then we replace I_L with $I_L + J_L^n$ so that $B_L \to C_L$ has nilpotent kernel.

Consider the set Ξ of pairs $\xi = (L, I_L, J_L)$ where $L \subseteq \Lambda$ is a finite subset, and $I_L \subseteq P_L$ and $J_L \subseteq Q_L$ are finitely generated ideals as in the previous paragraph. Then $(B_L \to C_L)_\xi$ is a filtered system of finite and finitely presented A-algebras with colimit $(B \to C)$ which satisfies the conditions of the proposition. $\square$

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