Algebraic stacks #9

Mar 27, 2024

§ More on quotient stacks: stabilizer, inertia

§ Deligne–Mumford stacks

§ Quotient stacks and equivariant geometry

$G \to S$ smooth group scheme $\curvearrowright X$ scheme or alg. space.

$$\begin{tikzcd} P \arrow[r, "\gamma"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d, "p"] \arrow[dr, phantom, "\square"] & S \arrow[d] \\ T \arrow[r, "(\alpha,\beta,\gamma)"'] & {[X/G]} \arrow[r] & BG \end{tikzcd}$$

universal G-torsor

$X \to [X/G]$ G-torsor $\Rightarrow$ behaves as if G acts freely.

geometry on stack $\mathcal{X} = [X/G]$	G-equivariant geometry on X
$\lvert\mathcal{X}\rvert$	G-orbits of $\lvert X\rvert$
$\mathrm{Aut}(x)$	$\mathrm{Stab}(x) \hookrightarrow G$
$\mathrm{QCoh}(\mathcal{X})$	$\mathrm{QCoh}^G(X)$
$\Gamma(\mathcal{X}, \mathcal{F})$	$\Gamma(X, p^*\mathcal{F})^G$

Let's make this more precise:

<u>Lemma</u>: $\mathcal{X}(\bar{k}) = X(\bar{k})/G(\bar{k})$

<u>proof</u>: Over an algebraically closed field, every G-torsor is <u>trivial</u>.

$$\begin{tikzcd} G_{\bar{k}} \cong P \arrow[r, "\gamma"] \arrow[d, "\beta"'] & X \arrow[d] \\ \operatorname{Spec} \bar{k} \arrow[r, "\alpha"'] & S \end{tikzcd}$$

$(\alpha, \gamma) \Longleftrightarrow$ choosing $\gamma(e) \in X(\bar{k})$

Isomorphism
$$\begin{tikzcd} G_{\bar{k}} \arrow[rr, "m_g"] \arrow[dr] & & G_{\bar{k}} \arrow[dl] \\ & \operatorname{Spec} \bar{k} & \end{tikzcd}$$
$\rightsquigarrow g \in G(\bar{k})$

$\square$

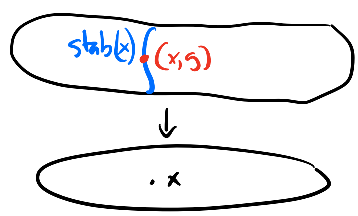
Def: $\mathrm{Stab}(X) = \{(g,x) : g.x = x\} \hookrightarrow G \times_S X$

$$\begin{tikzcd} \mathrm{Stab}(X) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & G \times_S X \arrow[d, "(\mathrm{pr}_2, \sigma)"] \\ X \arrow[r, "\Delta_X"] & X \times_S X \end{tikzcd}$$

Def: For $T \xrightarrow{x} X$, $\mathrm{Stab}(x) = \mathrm{Stab}(X) \times_X T \hookrightarrow G \times_S T$, group scheme $/T$.

A T'-point of $\mathrm{Stab}(x)$ is $g \in G(T')$ such that $g.x|_{T'} = x|_{T'}$.

__Picture:__



$\mathrm{Stab}(X)$ family of stabilizer group schemes

X

__Lemma:__ Given $T \xrightarrow{x} X$ have $\underline{\mathrm{Aut}}(p \circ x) = \mathrm{Stab}(x)$.

__proof:__ Have cartesian diagram:

$$\begin{tikzcd} \mathrm{Stab}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "x"] \\ \mathrm{Stab}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "p"] \\ I_{\mathcal{X}} \arrow[r] & \mathcal{X} \end{tikzcd}$$

$\square$

__Rmk:__ $T \xrightarrow{y} \mathcal{X} = [X/G]$ only gives group scheme $\mathrm{Aut}(y) \to T$.

Lifting to $T \xrightarrow{x} X \xrightarrow{p} \mathcal{X}$ gives __embedded__ $\mathrm{Aut}(p \circ x) = \mathrm{Stab}(x) \hookrightarrow G$.

Different lift $T \xrightarrow{x'} X \to \mathcal{X}$ gives __different embedding__. If $x' = g.x$ then $\mathrm{Stab}(x') = g\,\mathrm{Stab}(x)\,g^{-1}$!

__Rmk:__ $\mathrm{Stab}(X) \hookrightarrow G \times X$. One can show that

$$I_{\mathcal{X}} = [\mathrm{Stab}(X)/G] \hookrightarrow [G \times X / G]$$

where G-action on $G \times X$ is $g.(h,x) = (ghg^{-1}, gx)$ and $\mathrm{Stab}(X)$ is stable under G-action:

$$h.x = x \Rightarrow ghg^{-1}.gx = gx$$

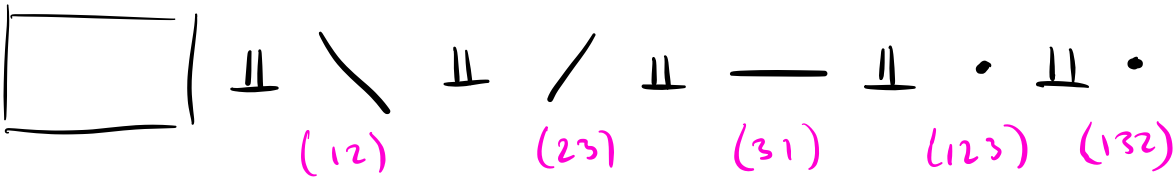
__SLOGAN:__ $\mathcal{X} = [X/G]$ remembers stabilizer groups, but only as abstract groups, not as subgroups of G.

Ex: $S_3 \curvearrowright \{x+y+z=0\} \hookrightarrow \mathbb{A}^3$ by permutation

$$\{x+y+z=0\} \cong \mathbb{A}^2$$

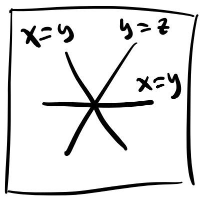
$\mathbb{A}^2 \times S_3$ ↺

$\mathrm{Stab}(\mathbb{A}^2) =$

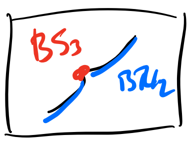


$\big\downarrow$ (deg 1–6)

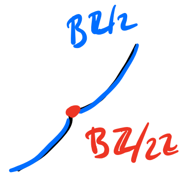
$\mathbb{A}^2 =$



$I_{\mathfrak{X}} = [\mathrm{Stab}(\mathbb{A}^2)/S_3] =$



$\sqcup$

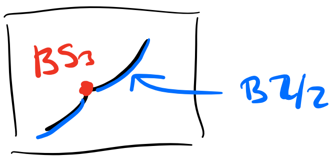


(deg 1–3)

$\sqcup \; \bullet \; B\mathbb{Z}/3\mathbb{Z}$ (deg 2)

$\downarrow$

$\mathfrak{X} = [\mathbb{A}^2/S_3] =$



$S_3 \curvearrowright S_3$ by conjugation: orbits $\{e\}, \{(12)\}, \{(123)\}$

size $1, \; 3, \; 2$

stabilizers $S_3 \quad \mathbb{Z}/2\mathbb{Z} \quad \mathbb{Z}/3\mathbb{Z}$

$$\begin{array}{ccc} \{e\} \cup \{(12),(23),(31)\} \cup \{(123),(132)\} = S_3 & \longrightarrow & [S_3/S_3] = BS_3 \sqcup \underset{(\text{deg } 3)}{B\mathbb{Z}/2\mathbb{Z}} \sqcup \underset{(\text{deg } 2)}{B\mathbb{Z}/3\mathbb{Z}} \\ \downarrow & & \downarrow \text{deg } 6 \\ * & \longrightarrow & BS_3 \end{array}$$

§ Deligne–Mumford stacks

Def (8.3.1) A stack $\mathcal{X}$ is **Deligne–Mumford** if

(1) $\Delta_{\mathcal{X}}$ is representable ← (recall, means $S \times_{\mathcal{X}} T$ algebraic space $\forall S, T$ sch)

(2) $\underbrace{\exists U \longrightarrow \mathcal{X} \text{ étale surjective with } U \text{ scheme}}_{\text{an étale presentation}}$

Ex: $G \curvearrowright X \Rightarrow X \xrightarrow{p} [X/G]$ G-torsor

- G smooth $\Rightarrow$ p smooth $\Rightarrow$ $[X/G]$ algebraic stack
- G étale $\Rightarrow$ p étale $\Rightarrow$ $[X/G]$ DM-stack.

Thm (8.3.3) $\mathcal{X}$ algebraic stack. TFAE

(1) $\mathcal{X}$ is DM.

(2) $\Delta_{\mathcal{X}}$ is unramified.

(3) $\forall x : \operatorname{Spec} k \longrightarrow \mathcal{X}$, $\operatorname{Aut}(x) \to \operatorname{Spec} k$ is étale.

Rmk:
- G usual group $\Rightarrow$ $\underline{G}_S = \coprod_{g \in G} S \to S$ constant group scheme $\Rightarrow$ étale
  $(S_n, \mathbb{Z}/n\mathbb{Z}, \dots)$ $\Rightarrow [X/G]$ DM
- G/S of $\dim \geq 1$ $\Rightarrow$ not étale $\Rightarrow$ BG not DM
  $(\mathbb{G}_a, \mathbb{G}_m, GL_n, \dots)$

Rmk: In char 0, all group schemes reduced so (loc) finite$/k$ $\Leftrightarrow$ étale$/k$.

In char p:
- $\mathbb{Z}/p\mathbb{Z}$ étale — $B\mathbb{Z}/p\mathbb{Z}$ DM
- μ_p, α_p finite, non-reduced — $B\mu_p$, $B\alpha_p$ not DM

Prop (8.4.14): $\mathcal{M}_g$ is DM for $g \geq 2$.

proof: Given $C \to \operatorname{Spec} k$ curve genus $g \geq 2$

need to prove $\operatorname{Aut}(C) \to \operatorname{Spec} k$ étale, or equiv, unramified.

Formally unramified: $\forall$ $A = A'/I$, $I^2 = 0$

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d] & \operatorname{Aut}(C) \arrow[d] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed, "\exists^{\leq 1}"] & \operatorname{Spec} k \end{tikzcd}$$

That is need to prove $\operatorname{Aut}(C_{A'}) \to \operatorname{Aut}(C_A)$ injective.

Lifts in diagram (non-canonically) bijective to

$$\begin{tikzcd} C_A \arrow[d] & C_L \arrow[l] \arrow[d] \arrow[dl, phantom, "\square"] \\ \operatorname{Spec} A & \operatorname{Spec} L \arrow[l] \end{tikzcd}$$

$$H^0(C_A, T_{C_A/A} \otimes_A I)$$

b/c C smooth. (relation b/w auto. and derivations)

This is zero, b/c $\forall \operatorname{Spec} L \to \operatorname{Spec} A$ have

$$H^0(C_L, (\text{—''—}) \otimes_A L) = \underbrace{H^0(C, T_C)}_{=0} \otimes_k (I \otimes_A L) = 0$$

b/c $\deg(T_C) = 2 - 2g < 0$ $\square$

Ex: $(g=0)$

$$\begin{tikzcd} \mathbb{P}^1 \arrow[d] & \\ \operatorname{Spec} k \arrow[r, "x"', "{[\mathbb{P}^1]}" below] & \mathcal{M}_0 \end{tikzcd}$$

$\operatorname{Aut}(x) = \operatorname{Aut}(\mathbb{P}^1/k) = PGL_2$ dim 3.

In fact, $\mathcal{M}_0 = BPGL_2$.

Ex: $(g=1)$

$$\begin{tikzcd} C \arrow[d] \\ \operatorname{Spec} \bar{k} \arrow[r, "x"] & \mathcal{M}_1 \end{tikzcd}$$

C genus 1 curve. Choose point $O \in C(\bar{k})$.
Then (C, O) elliptic curve and $\operatorname{Aut}(C, O)$ finite. (étale)
but $\operatorname{Aut}(C)$ infinite.

$$\begin{tikzcd} \operatorname{Aut}(C,O) \arrow[r, hook] & \operatorname{Aut}(C) & t_x = (p \mapsto p + x) \\ & C \arrow[u, hook] & x \arrow[u, mapsto] \end{tikzcd}$$

translation by x.

Proof of Thm 8.3.3:

(ii) $\Rightarrow$ (iii):

$$\begin{tikzcd} \operatorname{Aut}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d, "(x,x)"] \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}", "\text{unramified}"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

$\Rightarrow \operatorname{Aut}(x) \to \operatorname{Spec} k$ unramif $\Leftrightarrow$ étale over a field.

(iii) $\Rightarrow$ (ii): unramif $\Leftrightarrow$ fibers are étale $\Leftrightarrow$ geom fibers are étale

so consider

$$\begin{tikzcd} \operatorname{Isom}(x,y) \arrow[r] \arrow[d] & \operatorname{Spec} \bar{k} \arrow[d, "(x,y)"] \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

Need to prove $\operatorname{Isom}(x,y) \to \operatorname{Spec} k$ étale.

Either $\operatorname{Isom}(x,y) = \emptyset$ (ok!) or $\operatorname{Isom}(x,y)$ has a $\bar{k}$-point α b/c loc. of finite type and $\bar{k}$ alg closed.

Then
$$\operatorname{Aut}(x) \xrightarrow{\cong} \operatorname{Isom}(x,y) \xleftarrow{\cong} \operatorname{Aut}(y)$$
$$\varphi \longmapsto \alpha \circ \varphi \qquad \psi \circ \alpha \longmapsfrom \psi$$

So $\operatorname{Aut}(x)$ étale $\Rightarrow \operatorname{Isom}(x,y)$ étale.

(i) $\Rightarrow$ (ii): Let $U \to \mathcal{X}$ étale presentation. Then

$$\begin{tikzcd} U \times_{\mathcal{X}} U \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\square"] \arrow[rr, bend left, "\text{étale}"] & U \times U \arrow[r, "g"', "\text{repr.}"] \arrow[d, "\text{étale surj}"] & U \\ \mathcal{X} \arrow[r, "\Delta_{\mathcal{X}}"] & \mathcal{X} \times \mathcal{X} & \end{tikzcd}$$

$$\left.\begin{array}{l} g \circ f \text{ étale} \Rightarrow \text{unramified} \\ \Delta_g \text{ mono} \Rightarrow \text{unramified} \end{array}\right\} \Rightarrow f \text{ unramified}$$

(ii) $\Rightarrow$ (i): Let $U \xrightarrow{p} \mathcal{X}$ smooth presentation.

STEP 1: $\exists\, \Omega_{U/\mathcal{X}}$.

<u>Lemma</u>: $\mathcal{X} \xrightarrow{f} \mathcal{Y}$ representable map, $\mathcal{Y}$ algebraic stack. Then $\exists\, \Omega_{\mathcal{X}/\mathcal{Y}}$ quasi-coherent sheaf on $\mathcal{X}$. If f smooth, then $\Omega_{\mathcal{X}/\mathcal{Y}}$ locally free of finite rank.

<u>proof sketch</u>: Pick smooth pres $Y' \to \mathcal{Y}$. Then have

$$\begin{tikzcd} X'' \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\pi_2"'] \arrow[d] \arrow[dr, phantom, "\square"] & X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ Y'' := Y' \times_{\mathcal{Y}} Y' \arrow[r, shift left] \arrow[r, shift right] & Y' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

$$\pi_1^* \Omega_{X'/Y'} \underset{\text{can}}{\cong} \Omega_{X''/Y''} \underset{\text{can}}{\cong} \pi_2^* \Omega_{X'/Y'}$$

$\Rightarrow$ gluing data $\Rightarrow$ descends to $\Omega_{\mathcal{X}/\mathcal{Y}}$. $\square$

$$\begin{tikzcd} U \arrow[d, "p"] \\ \mathcal{X} \arrow[d] \\ S = \operatorname{Spec} \mathbb{Z} \end{tikzcd} \Rightarrow \quad \mathcal{O}_U \xrightarrow{d} \Omega_{U/S} \longrightarrow \Omega_{U/\mathcal{X}}$$

STEP 2: $\Omega_{U/S} \to \Omega_{U/\mathcal{X}}$ is surjective

(It's here we're using that $\Delta_{\mathcal{X}}$ unramified)

(a) Conceptual proof using cotangent complexes: (requires a lot of machinery for stacks!)

exact triangle
$$\mathbb{L}_{\mathcal{X}/S}|_U \to \mathbb{L}_{U/S} \to \mathbb{L}_{U/\mathcal{X}} \xrightarrow{+1}$$
$$\mathbb{L}_{\mathcal{X}/S}|_U \cong \mathbb{L}_{\Delta_{\mathcal{X}}}[-1]|_U$$

$\Delta_{\mathcal{X}}$ unramified $\Rightarrow$ $\mathbb{L}_{\Delta_{\mathcal{X}}}$ 1-connective $\Rightarrow$ $\mathbb{L}_{\mathcal{X}}$ 0-connective

$$\Rightarrow \quad \mathcal{H}^0(\mathbb{L}_{U/S}) \to \mathcal{H}^0(\mathbb{L}_{U/\mathcal{X}}) \to \underbrace{\mathcal{H}^1(\mathbb{L}_{\mathcal{X}/S}|_U)}_{=0}$$

(b) Explicit proof using presentation:

$$\begin{tikzcd}
 & U \times_{\mathcal{X}} U \arrow[r, "\pi_2"] \arrow[dl, "i"', "\text{unramified}"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & U \arrow[d, "p"] \\
U \times U \arrow[r, "\pi_1"] & U \arrow[r] & \mathcal{X}
\end{tikzcd}$$

(i unramified b/c pull-back of $\Delta_{\mathcal{X}}$)

$$\pi_2^* \Omega_{U/S} \to \pi_2^* \Omega_{U/\mathcal{X}}$$
$$\pi_2^* \Omega_{U/S} \cong i^* \Omega_{\pi_1}, \qquad \pi_2^* \Omega_{U/\mathcal{X}} \cong \Omega_q$$

$$i^* \Omega_{\pi_1} \longrightarrow \Omega_q \longrightarrow \underbrace{\Omega_i}_{=0} \longrightarrow 0 \qquad \text{exact seq.}$$

0 b/c i unramified.

STEP 3: $\exists$ factorization $p: U \xrightarrow{\text{étale}} \mathcal{X} \times \mathbb{A}^n \longrightarrow \mathcal{X}$ Zariski-locally on U.

Essentially same proof as when $U \to X$ smooth morphism of schemes:

- Pick $u \in |U|$.
- $(\Omega_{U/\mathcal{X}})_u$ free w/ basis $e_1, \dots, e_n$.
- $\operatorname{im}(d : \mathcal{O}_U \to \Omega_{U/S})$ locally generates $\Omega_{U/S}$.
- Since $\Omega_{U/S} \to \Omega_{U/\mathcal{X}}$ surjective, Zar-locally on U can write $e_i = df_i$

This gives $U \xrightarrow{f_1, \dots, f_n} \mathbb{A}^n$ after repl. U with some open nbhd of u.
$U \xrightarrow{f} \mathcal{X} \times \mathbb{A}^n$ is étale at u b/c:

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "p"', "\text{smooth}"'] & & \mathcal{X} \times \mathbb{A}^n \arrow[dl, "\text{smooth}"] \\ & \mathcal{X} & \end{tikzcd} \qquad \Omega_{\mathcal{X}\times\mathbb{A}^n/\mathcal{X}}\big|_U \longrightarrow \Omega_{U/\mathcal{X}}$$

iso at u by construction.

STEP 4: Slice presentation:

For simplicity assume $\mathcal{X} \to S = \operatorname{Spec} \bar{k}$ finite type and choose u in Step 3 to be a $\bar{k}$-point. Let z be the image of u under $U \to \mathbb{A}^n$. Then consider

$$\begin{tikzcd} V \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "\text{étale}"] \\ \mathcal{X} \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \times \mathbb{A}^n \arrow[d] \\ * \arrow[r, hook, "z"] & \mathbb{A}^n \end{tikzcd}$$

Then $V \to \mathcal{X}$ étale neighborhood of $x = p(u)$.

For every $x \in |\mathcal{X}|$ closed, $\exists u$ as above
$\Rightarrow$ get an étale presentation. $\square$