Algebraic stacks #8

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§ Algebraic stacks: def, Isom

§ Examples (quotient stacks, $\mathcal{M}_g$)

§ Definition

From this point: <u>stack</u> means

category fibered in groupoids $\mathcal{X} \longrightarrow Sch$ (or Sch/S)
with sheaf condition (:= effective descent) for étale topology

$$\Updownarrow$$

pseudo-2-functor $Sch^{op} \longrightarrow Grpd$ w/ sheaf condition

Recall: Stacks is a (2,1)-category (full subcategory of CFGs)

Def: A morphism of stacks $\mathcal{X} \longrightarrow \mathcal{Y}$ is **representable**
if $\forall T$ scheme, $\mathcal{X} \times_{\mathcal{Y}} T$ is an **algebraic space**

$\iff$ $\forall T$ <u>alg space</u>, ——— " ———

Lemma 8.1.3

Def: A stack $\mathcal{X}$ is **algebraic** (or an **Artin stack**) if
(i) $\Delta_{\mathcal{X}} : \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ is representable
(ii) $\underbrace{\exists\, U \longrightarrow \mathcal{X} \text{ smooth surjective with } U \text{ scheme}}_{\text{a smooth presentation}}$

alg. space
Rmk 8.1.11

Rmk As in def. of alg spaces, (i) $\Rightarrow$ $U \longrightarrow \mathcal{X}$ in (ii) is
representable. Indeed, given $U \longrightarrow \mathcal{X}$, $V \longrightarrow \mathcal{X}$ w/ U, V sch

$$\begin{tikzcd} U \times_{\mathcal{X}} V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U \times V \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \times \mathcal{X} \end{tikzcd} \implies U \times_{\mathcal{X}} V \text{ alg space.}$$

§Isom

(Lect 5, Prop 4.6.2)

<u>Recall</u>: given $x, y \in \mathcal{X}(T) \rightsquigarrow$ <u>Isom</u>(x,y) sheaf on $Sch_{/T}$

$\mathcal{X}$ being a prestack (= descent but not necc eff. descent)

$\iff$ <u>Isom</u>(x,y) sheaf $\forall T$.

<u>Lem 8.1.8</u>: Let $\mathcal{X}$ be a stack. TFAE

(a) $\Delta_{\mathcal{X}}$ representable

(b) $\forall T$ schemes, $\forall x, y \in \mathcal{X}(T)$: <u>Isom</u>(x,y) algebraic space.

$\iff \forall T$ alg. spaces ——— " ———

<u>pf</u>:

$$\begin{tikzcd} & T \arrow[d, "\alpha"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

- $\alpha \iff x, y \in \mathcal{X}(T)$
- Fiber product is <u>Isom</u>(x,y): (unravel definitions) □

<u>Rmk</u>: Can similarly define <u>Isom</u>(x,y) for any alg space T and $T \xrightarrow{x,y} \mathcal{X}$ or even alg stack T.

The set of T-points of <u>Isom</u>(x,y) is 2-isomorphisms:

$$\begin{tikzcd} T \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{X} \end{tikzcd}$$

§ (2,1)-category

Rmk: AlgStacks $\subset$ Stacks $\subset$ CFGs full (2,1)-subcategories

2-fiber products of CFGs are calculated pointwise:

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})(T) \simeq \mathcal{X}(T) \times_{\mathcal{Z}(T)} \mathcal{Y}(T) \quad \text{2-fiber prod of groupoids}$$

2-fiber products of stacks are stacks
(Stacks $\subset$ CFGs right 2-adjoint so preserves 2-limits)

Prop 8.1.6: 2-fiber products of algebraic stacks are algebraic.

That is, given

$$\begin{tikzcd} & \mathcal{Y} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{Z} \end{tikzcd}$$

diagram of algebraic stacks,

then 2-fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ as CFGs is an algebraic stack.

§ Aut

Special case of Isom:

Given $T \xrightarrow{x} \mathcal{X}$, let $\underline{\mathrm{Aut}}(x) := \underline{\mathrm{Isom}}(x, x)$.

If $\mathcal{X}$ algebraic (or merely $\Delta_{\mathcal{X}}$ representable) then $\underline{\mathrm{Aut}}(x)$ is an algebraic space + sheaf of groups $/T$.

"group algebraic space"

Thm (Artin) Any group algebraic space / field k is a group scheme.

$\Rightarrow \forall\, \mathrm{Spec}\, k \to \mathcal{X}$, $\mathrm{Aut}(x)$ group scheme $/k$.

Rmk: If $\mathcal{X}$ fibered in setoids (i.e. sheaf of sets), then $\underline{\mathrm{Aut}}(x)$ trivial $\forall x$. (whereas $\underline{\mathrm{Isom}}(x, y) \subset T$ monomorphism)

Def: $|\mathcal{X}| = \{\mathrm{Spec}\, k \to \mathcal{X}\} / \sim$ as before

+ topology as before. $\rightsquigarrow$ functor: $\mathrm{Stacks} \longrightarrow \mathrm{Top}$

Warning: For $x \in |\mathcal{X}|$ don't get group $\mathrm{Aut}(x)$.

But if $\mathrm{Spec}\, k \xrightarrow{x} \mathcal{X}$, $\mathrm{Spec}\, k' \xrightarrow{x'} \mathcal{X}$ and $x \sim x'$ then

$$\mathrm{Aut}(x) \times_k \mathrm{Spec}\, L \cong \mathrm{Aut}(x') \times_k \mathrm{Spec}\, L$$

if
$$\begin{tikzcd} & \mathrm{Spec}\, k \arrow[dr, "x"] \arrow[d, phantom, "\circ"] & \\ \mathrm{Spec}\, L \arrow[ur] \arrow[r] & \mathrm{Spec}\, k' \arrow[r, "x'"] & \mathcal{X} \end{tikzcd}.$$

§ Inertia stack

Def (8.1.17) The inertia stack $I_{\mathcal{X}}$ is the fiber product

$$\begin{tikzcd} I_{\mathcal{X}} \arrow[r] \arrow[d] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

a "group algebraic space over $\mathcal{X}$".

Rmk:

$$\begin{tikzcd} \mathrm{Aut}(x) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "x"] \arrow[dd, bend left, "{(x,x)}"] \\ I_{\mathcal{X}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\Delta"] \\ \mathcal{X} \arrow[r, "\Delta"'] & \mathcal{X} \times \mathcal{X} \end{tikzcd}$$

So $I_{\mathcal{X}} \to \mathcal{X}$ the "universal family of automorphisms".

Explicitly: $I_{\mathcal{X}}(T)$ groupoid with

- objects: $x \in \mathcal{X}(T)$, $g \in \mathrm{Aut}(x)$
- morphism $(x, g) \to (y, h)$ is $x \xrightarrow{f} y \in \mathcal{X}(T)$ s.th:

$$\begin{tikzcd} x \arrow[r, "g"] \arrow[d, "f"'] \arrow[dr, phantom, "\circ"] & x \arrow[d, "f"] \\ y \arrow[r, "h"'] & y \end{tikzcd}$$

(diagram in groupoid $\mathcal{X}(T)$)

§ Free actions

Recall: G étale (e.g. finite constant) acting freely on X, that is, action $G \times X \xrightarrow{\sigma} X$ s.th. $G \times X \overset{(pr_2, \sigma)}{\hookrightarrow} X \times X$ equivalence relation $\Rightarrow$ $X \xrightarrow{q} X/G$ sheaf quotient

Coequalizer: $G \times X \rightrightarrows X \xrightarrow{q} X/G$, cartesian square:

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "pr_2"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ X \arrow[r, "q"] & X/G \end{tikzcd}$$

Then $X \to X/G$ G-torsor ($G \curvearrowright X$ and étale-locally, namely along q itself, becomes trivial)

For any $T \to X/G$ get:

$$\begin{tikzcd} P \arrow[r, "G\text{-equi}"] \arrow[d, "G\text{-torsor}"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "q"] \\ T \arrow[r] & X/G \end{tikzcd}$$

For $y \in |X/G|$ get (assuming X/G quasi-separated)

$$\begin{tikzcd} q^{-1}(y) = P \arrow[r, hook] \arrow[d, "G\text{-orbit}"'] & X \arrow[d, "q"] \\ \operatorname{Spec} \kappa(y) \arrow[r, hook] & X/G \end{tikzcd}$$

Thm (Artin) If G smooth $\curvearrowright X$ freely, then sheaf quotient (étale top) X/G alg. space.

If G flat $\curvearrowright X$ freely, then fppf-sheaf quotient alg. space.

§ Quotient stacks (ex 8.1.12)

Let $G \to S$ smooth group scheme $\curvearrowright X \xrightarrow{\pi} S$ scheme/alg. space.

The **stack quotient** $[X/G]$ is the following CFG.

$\boxed{\text{Objects}}$ diagrams

$$\begin{tikzcd} P \arrow[r, "\gamma"] \arrow[d, "\beta"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "\pi"] \\ T \arrow[r, "\alpha"] & S \end{tikzcd} = \text{triples } (\alpha, \beta, \gamma)$$

where
- T scheme
- $P \xrightarrow{\beta} T$ G_T-torsor ($\Rightarrow$ P scheme if $G \to S$ affine; o/w algebraic space)
- $P \xrightarrow{\alpha} X$ G-equivariant

$\boxed{\text{Morphisms}}$ $(f, g) : (\alpha', \beta', \gamma') \to (\alpha, \beta, \gamma)$

diagrams

$$\begin{tikzcd} P' \arrow[r, "g"] \arrow[d, "\beta'"'] \arrow[dr, phantom, "\square"] \arrow[rr, bend left, "\gamma'"] & P \arrow[r, "\gamma"] \arrow[d, "\beta"] \arrow[dr, phantom, "\circ"] & X \arrow[d, "\pi"] \\ T' \arrow[r, "f"] \arrow[rr, bend right, "\alpha'"'] & T \arrow[r, "\alpha"] & S \end{tikzcd}$$

s.th.
- g G-equivariant
- square cartesian
- triangles commute

Structure map

$$\begin{array}{ccccc} [X/G] & \longrightarrow & Sch/_S & \longrightarrow & Sch \\ (\alpha, \beta, \gamma) & \longmapsto & T \to S & \longmapsto & T \\ (f, g) & \longmapsto & f & \longmapsto & f \end{array}$$

(ex 8.1.12)

<u>Thm</u>: $[X/G]$ is an algebraic stack.

<u>pf</u>: (0) $[X/G]$ is a CFG. Equivalently, every map is cartesian.
Follows from cartesian square in definition:

$$\begin{tikzcd}
P'' \arrow[r, dashed] \arrow[d] \arrow[rr, bend left, "\circ"] \arrow[rrr, bend left=40] & P' \arrow[r] \arrow[d] \arrow[dr, phantom, "D"] & P \arrow[r] \arrow[d] & X \arrow[d] \\
T'' \arrow[r] \arrow[rr, bend right, "\circ"'] & T' \arrow[r] & T \arrow[r] & S
\end{tikzcd}$$

(1) $[X/G]$ is a stack: let $\{T_i \to T\}$ étale covering in Sch

$$((\alpha_i, \beta_i, \gamma_i), \text{ descent datum}) \in [X/G](\{T_i \to T\})$$

(1a) Since S is a sheaf, we can glue the $\alpha_i : T_i \to S$ to $\alpha : T \to S$.

(1b) Since G-torsors is a stack (Exc 4.G) we can glue the $P_i \xrightarrow[\beta_i]{} T_i$ to a torsor $P \xrightarrow[\beta]{} T$.

(1c) Since X is a sheaf, we can glue the $\gamma_i : P_i \to X$ to $\gamma : P \to X$.

G-equivariance means:
$$\begin{tikzcd}
G \times P \arrow[r] \arrow[d, "\text{action}"] & G \times X \arrow[d, "\text{action}"] \\
P \arrow[r] & X
\end{tikzcd}$$
commutes

Commutativity can be checked locally on $G \times P$ since X sheaf.

(2) $\Delta_{[X/G]}$ is representable.

Let $T \xrightarrow{x_1, x_2} [X/G]$, i.e., for $i = 1, 2$.

$$\begin{tikzcd} P_i \arrow[r, "\gamma_i"] \arrow[d, "\beta_i"'] & X \arrow[d] \\ T \arrow[r, "\alpha_i"'] & S \end{tikzcd}$$

Need to prove $\underline{\mathrm{Isom}}(x_1, x_2) \in Sh(Sch_{/T})$ is algebraic space.

$$\begin{tikzcd} \underline{\mathrm{Isom}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & W \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d] \\ {[X/G]} \arrow[r] & {[X/G] \times_S [X/G]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/G] \times [X/G]} \arrow[d] \\ & S \arrow[r] & S \times S \end{tikzcd}$$

W scheme b/c S scheme. It is the loc. closed subscheme of T where $\alpha_1 = \alpha_2$. So WLOG $\alpha_1 = \alpha_2$.
(Olsson treats $[X/G]$ as stack over S and thus $\alpha_1 = \alpha_2$ by definition).

By Exc 5.G, that $\underline{\mathrm{Isom}}/T$ algebraic space is étale-local on T.

$\Rightarrow$ WLOG $P_1 \to T$, $P_2 \to T$ $\underline{\underline{\text{trivial}}}$ torsors

Fix $P_i \cong G_T$. G_T-equi. isom. $P_1 \cong P_2$ is then given by mult. by $g \in G(T)$ so

$$\underline{\mathrm{Isom}}(x_1, x_2)(T' \to T) = \{ g \in G(T') : \gamma_1 = \gamma_2 \circ m_g \}$$

mult. by g.

$$\begin{tikzcd}
 & & X \\
G_{T'} \arrow[rru, bend left, "\gamma_1"] \arrow[r, "m_g"] \arrow[dr, "\mathrm{proj}"'] & G_{T'} \arrow[ru, "\gamma_2"'] \arrow[d, "\mathrm{proj}"] & \\
 & T' &
\end{tikzcd}$$

Commutativity of diagram $\iff \gamma_1(e) = \gamma_2(g)$ so:

$$\begin{tikzcd}
\underline{\mathrm{Isom}}(x_1, x_2) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & G_T \arrow[d, "{(\gamma_1(e),\, \gamma_2(-))}"] \\
X \arrow[r, "\Delta_X"] & X \times X
\end{tikzcd}$$

is cartesian.

X algebraic space $\Rightarrow \Delta_X$ schematic $\Rightarrow \underline{\mathrm{Isom}}(x_1, x_2)$ scheme.

So has shown:

- P_1, P_2 trivial $\Rightarrow$ Isom scheme.

For P_1, P_2 general, Isom only an algebraic space.

(3) $\exists$ smooth presentation. We have a map

$$X \xrightarrow{P} [X/G]$$

given by:

$$\begin{tikzcd} G \times X \arrow[r, "\gamma"] \arrow[d, "\beta = pr_2"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "\pi"] \\ X \arrow[r, "\alpha = \pi"'] & S \end{tikzcd}$$

where α is the structure map π
β is the trivial torsor
γ is the G-action on X

P is smooth and surj: Pick $T \xrightarrow{(\alpha,\beta,\gamma)} [X/G]$:

$$\begin{tikzcd} I := \underline{\mathrm{Isom}}(P, (\alpha,\beta,\gamma)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "P"] \\ T \arrow[r, "(\alpha,\beta,\gamma)"'] & [X/G] \end{tikzcd}$$

Claim: $I \longrightarrow T$ is isomorphic to $P \xrightarrow{\beta} T$

Follows from
$$\begin{tikzcd} G_T \arrow[rr, "\cong", "f"'] \arrow[dr] & \circ & P \arrow[dl] \\ & T & \end{tikzcd}$$
isom of G-torsors $\iff$ section $T \xrightarrow{s} P$

$(f(e) = s\ ,\ f(g) = g \cdot s)$ $\square$

§ $\mathcal{M}_g$ (8.4.3)

$$\mathcal{M}_g(T) = \left\{ \begin{array}{l} C \xrightarrow{f} T \quad \text{proper, flat (of finite presentation)} \\ \quad \text{s.th. } C_t \text{ smooth geom. connected genus } g \ \forall t \in T \end{array} \right\}$$

<u>Tricanonical embedding</u>: $\omega_{C/T}^{\otimes 3}$ very ample relative to T, i.e.

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr] & \mathbb{P}\Big(\underbrace{f_* \omega_{C/T}^{\otimes 3}}_{\text{loc free of rank } N = 5g-5 \text{ but not nec free}}\Big) \arrow[d] \\ & T \end{tikzcd} \quad \text{closed embedding}$$

Let $\widetilde{\mathcal{M}}_g$ following stack:

$$\widetilde{\mathcal{M}}_g(T) = \left\{ C \xrightarrow{f} T \text{ as above } + f_* \omega_{C/T}^{\otimes 3} \underset{\cong}{\xrightarrow{\varphi}} \mathcal{O}_T^N \right\}$$

Then $\mathcal{M}_g = [\widetilde{\mathcal{M}}_g / GL_N]$ (action via postcomposing φ w/ $g \in GL_n$)

<u>Claim</u>: $\widetilde{\mathcal{M}}_g$ is a quasi-projective scheme.

To see this, first consider $\mathrm{Hilb}(\mathbb{P}^N)$ and the subfunctor $\mathcal{H}$:

$$\mathcal{H}(T) = \left\{ \begin{tikzcd} C \arrow[r, hook] \arrow[dr, "f"'] & \mathbb{P}^N \times T \arrow[d] \\ & T \end{tikzcd} : C \xrightarrow{f} T \text{ as in } \mathcal{M}_g \right\}$$

$$\subset \mathrm{Hilb}^P(\mathbb{P}^N)(T)$$

Hilbert polynomial: $P = (6g-6)T + 1 - g$

<u>Exc</u>: This is an open subfunctor. ($\Rightarrow \mathcal{H}$ quasi-projective)

Have $\widetilde{\mathcal{M}}_g \longrightarrow \mathcal{H}$. The difference is that in $\mathcal{H}$ we have line bundles $\omega_{C/T}^{\otimes 3}$, $\mathcal{O}_{\mathbb{P}^N}(1)|_C$ on C that need not agree whereas in $\widetilde{\mathcal{M}}_g$ canonically isomorphic.

$$\widetilde{\mathcal{M}}_g(T) \cong \left\{ \begin{tikzcd} C \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^N \times T \arrow[d] \\ & T \end{tikzcd} : \begin{array}{l} C \xrightarrow{f} T \text{ as in } \mathcal{M}_g, \ i \text{ closed immersion} \\ \omega_{C/T}^{\otimes 3} \underset{\psi}{\cong} i^* \mathcal{O}_T(1) \end{array} \right\}$$

Can show $\widetilde{\mathcal{M}}_g \longrightarrow \mathcal{H}$ affine. (In fact, a $\mathbb{G}_m$-torsor over its image.)

Olsson shows factors as $\widetilde{\mathcal{M}}_g \xhookrightarrow{\text{closed}} \widetilde{\mathcal{H}} \xrightarrow{\text{affine}} \mathcal{H}$.

Thus $\widetilde{\mathcal{M}}_g$ quasi-projective. $\square$

<u>Conclusion</u>: $\mathcal{M}_g = [\underset{\substack{\uparrow \\ \text{quasi-proj}}}{\widetilde{\mathcal{M}}_g} / GL_N]$ so algebraic stack.