Algebraic stacks #7

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§ Properties for algebraic spaces
§ Fiber products
§5.4

§ Points and topology
§ Quotients by non-free actions
§6

§ Properties: alg spaces

Def: Let P be a étale-local (=stable) property. We say that an alg space X has P if

(a) $\exists U \to X$ étale presentation: U has P

$\Updownarrow$ simple exc

(b) $\forall$ ——————— " ———————

Ex: P = regular, normal, reduced, loc. noetherian, ...

§ Properties: rep maps

Def: Let P be a property of morphisms of schemes, local on target (=stable). If $f: X \to Y$ is a morphism of alg. spaces (or sheaves) representable by schemes, then f has P if

Def 5.4.3 → (a) $\exists T \to Y$ étale pres.: $X \times_Y T \to T$ has P

$\Updownarrow$

(b) $\forall$ ——————— " ———————

$\Updownarrow$

(c) $\forall T \to Y$, T scheme, —— " ————

simple exc.

only makes sense for alg. spaces

definition when X, Y sheaves (5.1.5)

Ex: P = closed imm, open imm, loc cl. immersion, affine, quasi-affine, quasi-compact, proper, separated, q separated, ...

Def: $f: X \to Y$ is closed/open/loc cl. immersion, affine, quasi-affine, if f is representable by schemes and ——— " ———

(5.4.6)
Rmk: X alg. space $\Rightarrow$ Δ_X rep. by schemes by def.

$f: X \to Y$ mor of alg sp $\Rightarrow$ $\Delta_{X/Y}$ rep by schemes b/c

$$\begin{tikzcd} X \arrow[r] \arrow[rr, bend left, "\text{repr. by schemes}"] & X \times_Y X \arrow[r, "\text{mono}"'] & X \times X \end{tikzcd} \Rightarrow \underset{\text{repr. by schemes}}{X \to X \times_Y X}$$

Def: An alg space X is:

(a) separated if Δ_X is a closed immersion

(b) locally separated if Δ_X is a locally closed immersion

(c) quasi-separated if Δ_X is quasi-compact.

(similarly for $f: X \to Y$ and $\Delta_{X/Y}$)

Rmk: Schemes are always locally separated.

Locally noetherian schemes are always quasi-separated.

(b/c open imm $\Rightarrow$ qcompact for locally noetherian)

Ex: $\mathbb{A}^\infty = \operatorname{Spec} k[x_1, x_2, x_3, \ldots]$. $\qquad \mathbb{A}^\infty \underset{\mathbb{A}^\infty \setminus 0}{\cup} \mathbb{A}^\infty$ not qsep.

Exc: Which of the examples on the last lecture are locally separated / quasi-separated?

§ Properties: non-rep. maps

Def: Let P be a property of morphisms of schemes that is local on base and source. We say that a morphism of algebraic spaces $f: X \longrightarrow Y$ is P if

(a) $\exists$

$$\begin{tikzcd} U \times_Y V \arrow[r] \arrow[d] \arrow[dd, bend right, "g"'] \arrow[dr, phantom, "\square"] & U \arrow[d, "\text{étale pres}"] \\ X \times_Y V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d] \\ V \arrow[r, "\text{étale pres}"'] & Y \end{tikzcd}$$

s.th. g is P

$\Updownarrow$

(a') $\forall$ ——————— " ———————

$\Updownarrow$

(b) $\exists$

$$\begin{tikzcd} U \arrow[r, "\text{étale pres}"] \arrow[dr, "g"'] & X \times_Y V \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ & V \arrow[r, "\text{étale pres}"'] & Y \end{tikzcd}$$

s.th. g is P

$\Updownarrow$

(b') $\forall$ —— " —— g is P.

Ex: P = loc. of finite type, loc. of finite presentation, flat, smooth, étale, surjective.

Prop 5.4.12: $\mathrm{AlgSp} \subset \mathrm{Shv}(\mathrm{Sch}_{\text{ét}})$ is closed under finite limits ($\Leftrightarrow$ fiber products + final object)

(used implicitly in (b)/(b') above)

proof (idea): If $U \to X$, $V \to Y$ étale pres, then

$$\begin{tikzcd}
U \times_S V \arrow[r] \arrow[d, "\text{étale pres}"] \arrow[dr, phantom, "\lrcorner"] & U \times V \arrow[d, "\text{étale pres}"] \\
X \times_S Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & X \times Y \arrow[d] \\
S \arrow[r, "\text{repr.}"'] & S \times S
\end{tikzcd}$$

Remains to prove $\Delta_{X \times_S Y}$ repr. ...

Want to define $f: X \to Y$ is P when P not local on source and f not repr. by schemes. Done case by case depending on P.

Def: An alg. space X is quasi-compact if $\exists\, U \to X$ étale pres with U qcompact (or $\Leftrightarrow$ affine)

X is noetherian if locally noetherian + qcompact + qseparated.

Def: $f: X \to Y$ morphism of alg. spaces is quasi-compact if $X \times_Y T$ quasi-compact for $\forall\, T \to Y$ w/ T qcpt scheme

$\Leftrightarrow \forall\, T \to Y$ étale w/ T qcpt sch.

$\Leftrightarrow T = V_i$ if $\coprod V_i \to Y$ étale pres

⋮

Def: $f: X \to Y$ proper if

- universally closed — define later
- quasi-compact
- locally of finite type
- separated

(quasi-compact + locally of finite type) = finite type

§ "Underlying" topological space

We will define:

$$\begin{array}{ccc} \mathrm{AlgSp} \subset \underset{\substack{\shortparallel \\ \mathrm{Shv}(\mathrm{Sch}_{et})}}{\mathrm{Spaces}} & \longrightarrow & \mathrm{Top} \\ X & \longmapsto & |X| \end{array}$$

<u>Def</u>: Let $X \in$ Spaces. As a <u>set</u>:

$$|X| = \{ \operatorname{Spec} k \xrightarrow{x} X : k \text{ field} \} / \sim$$

where $\operatorname{Spec} k \xrightarrow{x} X$, $\operatorname{Spec} k' \xrightarrow{y} X$ are equiv. if $\exists$

$$\begin{tikzcd} & \operatorname{Spec} k \arrow[dr, "x"] & \\ \operatorname{Spec} L \arrow[ur] \arrow[dr] \arrow[rr, phantom, "\circ"] & & X \\ & \operatorname{Spec} k' \arrow[ur, "y"'] & \end{tikzcd}$$

<u>Rmk</u>: If X <u>scheme</u>, then $\exists$ minimal rep. $\operatorname{Spec} \kappa(x) \xhookrightarrow{\text{mono}} X$ $\forall x \in |X|$.

<u>Rmk</u>: $|-|$ is a functor.

Lemma: $|-|:$ Spaces $\to$ Set preserves colimits.

proof: • Coproducts preserved since $(\coprod X_i)(\operatorname{Spec} k) = \coprod X_i(\operatorname{Spec} k)$

• Coequalizers: if $R \rightrightarrows U \to X$ coequalizer in Spaces, then X sheafification of $T \mapsto \operatorname{coeq}(R(T) \rightrightarrows U(T))$

If $\bar{k}$ alg. closed, $\nexists$ non-trivial étale covers of $\operatorname{Spec} \bar{k}$

$\Rightarrow X(\operatorname{Spec} \bar{k}) = \operatorname{coeq}(R(\operatorname{Spec} \bar{k}) \rightrightarrows U(\operatorname{Spec} \bar{k}))$

$\Rightarrow |R| \rightrightarrows |U| \to |X|$ coequalizer. $\square$

Topology:

Def: $U \subset |X|$ open if $\exists j : V \to X$ open immersion such that $U = |j|(|V|)$.

Alternative def: $U \subset |X|$ open if $\forall$ schemes T and $T \xrightarrow{f} X$ $|f|^{-1}(U)$ is open.

Rmk: • This defines a topology on $|X|$.

• $f : X \to Y \Rightarrow |f| : |X| \to |Y|$ continuous

(because given $U \subset |Y|$ and j as above we have $f^{-1}(U) = $ image of $|V \times_Y X| \to |X|$.)

• Gives functor Spaces $\to$ Top.

• This functor preserves colimits. (Enough to prove that if $U \to X$ epimorphism then $|U| \to |X|$ submersive. Follows from $T' \to T$ étale covering $\Rightarrow |T'| \to |T|$ submersive.)

Def: $f : X \to Y$ closed if $|f|$ closed.

Universally closed if $\forall \underset{\text{sch.}}{T} \to Y$, $X \times_Y T \to T$ closed.

§ Main results

<u>Prop 6.3.4</u>: X quasi-separated algebraic space. Then every $x \in |X|$ is represented by unique monomorphism $\operatorname{Spec} \kappa(x) \hookrightarrow X$.
called the residue field

<u>Thm 6.4.1</u>: X quasi-separated algebraic space. Then $\exists\, U \subset X$ dense open such that U is a scheme.

<u>Thm</u> (Raynaud–Gruson '71, 5.7.6) X qcompact and qsep. alg. space. There $\exists$

$$\emptyset = U_0 \subset U_1 \subset U_2 \subset \cdots \subset U_n = X$$

filtration by qcpt open subspaces such that strata $U_k \setminus U_{k-1}$ are quasi-affine schemes. More precisely $\forall n$:

$$\exists\, U_n' \xrightarrow{p_n} U_n \quad \text{étale presentation with } U_n' \text{ (quasi-)affine}$$
$$p_n|_{U_n \setminus U_{n-1}} \text{ isomorphism.}$$

<u>Rmk</u>: • $X' := \coprod_{n=1}^{n} U_n' \to X$ is a Nisnevich presentation: Étale presentation such that every $\operatorname{Spec} k \to X$ lifts to X'.

- 6.3.4 + 6.4.1 easily follows from RG's thm. (free)
- Key to prove all 3 results is to study <u>quotients of finite group actions</u>
- All 3 results false w/o quasi-separated assumption:
  $\exists\, X$ alg space, $|X| = \{x\}$, X not a scheme, $\nexists\, \kappa(x)$.

(see counter-example from last lecture)

§ Quotients of finite actions (non nec. free)

G finite group $\curvearrowright X \Rightarrow$ action groupoid $\underset{\coprod_{g\in G} X}{G \times X} \underset{\rho}{\overset{\pi_2}{\rightrightarrows}} X$ finite étale

Thm (Grothendieck FGA Exp 212, Gabriel SGA3 Exp V, $+\varepsilon$)

Let $R \underset{t}{\overset{s}{\rightrightarrows}} U$ finite flat groupoid of affine schemes.

Let $U \xrightarrow{q} X$ coequalizer in category of affine schemes
i.e., $\Gamma(X, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_U) \rightrightarrows \Gamma(R, \mathcal{O}_R)$ equalizer. Then:

(1) $|R| \rightrightarrows |U| \to |X|$ coequalizer of top. spaces.
(2) $\mathcal{O}_X \to q_*\mathcal{O}_U \rightrightarrows q_* s_* \mathcal{O}_R$ equalizer

] q is a geometric quot

(3) q coequalizer in category of algebraic spaces $\leftarrow$ q is a categorical quot.
(4) q is integral.

Rmk: If $R \rightrightarrows U$ equivalence relation ($R \to U \times U$ mono)
then sheaf quotient is an algebraic space (ess. by definition).
By (3), coincides with $q: U \to X$ of theorem so in this case
q coequalizer in category of spaces (= sheaves).

In general (groupoid, not eq. rel.) sheaf quotient not algebraic space.
(But stack quotient is an algebraic stack).

Cor: Let $R \overset{s}{\underset{t}{\rightrightarrows}} U$ finite flat groupoid of schemes s.th.

(*) $\forall u \in U$ $\underbrace{\text{orbit}(u)}_{:= s(t^{-1}(u)) = t(s^{-1}(u))}$ is contained in an affine open subscheme

Then $\exists q: U \longrightarrow X$ w/ X scheme satisfying (1)–(4) above.

Rmk: • (*) neccessary for quotient with X scheme.

• (*) holds if U quasi-projective.