Algebraic stacks #6

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§ Algebraic spaces

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* Properties (local source/target, representable)
* Definition
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* Examples

§ Idea

Today: only sheaves of sets, no groupoids.

Given scheme X, open covering $U := \amalg U_i \to X$, obtain coequalizer

$$U \times_X U \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} U \longrightarrow X$$

in category of schemes. Also coequalizer in big Zariski site $Sh_{Set}(Sch_{Zar})$ — this is the sheaf condition for h_T for a scheme T.

Def: Let $\mathcal{C}$ be a category. An effective epimorphism in $\mathcal{C}$ is an epimorphism $X \longrightarrow Y$ s.th. $X \times_Y X \rightrightarrows X \longrightarrow Y$ is a coeq.

Ex: $U \to X$ effective epimorphism in Sch and $Sh_{Set}(Sch, Zar/Ét/fppf)$.

Fact: In a topos, every epimorphism is effective.

Can reconstruct X as $\mathrm{coeq}(U \times_X U \rightrightarrows U)$. The groupoid $U \times_X U \rightrightarrows U$ is an equivalence relation: $U \times_X U \longrightarrow U \times U$ is a monomorphism.

Idea of alg. spaces: Given étale equiv. rel. $R \rightrightarrows U$, take coequalizer X in $Sh(Sch_{ét})$. Then $U \longrightarrow X$ epi ($\Rightarrow$ eff. epi) and $R = U \times_X U$. Call sheaves such as X for algebraic spaces. Not always schemes...

Ex: discrete $G \curvearrowright U$ freely $\rightsquigarrow$ action groupoid $G \times U \rightrightarrows U$ equiv rel $\rightsquigarrow$ coequalizer is an algebraic space denoted U/G.

§ Properties [O, 5.1]

Consider $\mathcal{C} = (Sch_{/S})_E$ = $Sch_{/S}$ with étale topology (big étale site)

A property P of schemes is "stable" (or local for étale topology) if for any covering $\{U_i \to X\}$, if $\forall U_i$ has P $\Rightarrow$ X has P.

Ex: Locally noetherian, reduced, normal, regular are stable properties.

Let P be a property of morphisms of schemes that is

(i) stable under composition: f, g P $\Rightarrow$ $g \circ f$ P

(ii) stable under base change: f P $\Rightarrow$ f' P

$$\left(\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ Y' \arrow[r, "\text{any}"'] & Y \end{tikzcd}\right)$$

(iii) contains all isomorphisms.

We say that P is "stable" (or local on base/target) if given

$$\begin{tikzcd} X \times_Y U_i \arrow[r] \arrow[d, "f_i"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ U_i \arrow[r] & Y \end{tikzcd}$$

such that $\{U_i \to Y\}$ covering, then $\forall i$: f_i is P $\Rightarrow$ f is P.

We say that P is local on the source if given

$$\begin{tikzcd} U_i \arrow[r, "g_i"] & X \arrow[d, "f"] \\ & Y \end{tikzcd}$$

such that $\{U_i \to X\}$ covering, then $\forall i$: $f \circ g_i$ is P $\Rightarrow$ f is P.

(Usually also assume P local on target)

Ex: proper, quasi-compact, separated, quasi-separated, affine, quasi-affine are local on target. Also any property on geom fibers (surj, univ inj, geom conn fibers, ...)

Ex: loc. of finite type, loc. of finite pres., smooth, flat, étale, loc. quasi-finite are local on source + target.

Def [0, 5.1.5] Let $f : F \to G$ morphism of sheaves in big étale site. We say that f is representable by schemes (or schematic) if

$$\forall \text{scheme } T, \quad \forall T \to G, \qquad F \times_G T \text{ is a scheme}$$

If in addition, P is a stable property of morphisms, we say that f is P if $\forall T \to G$, $f_T : F \times_G T \to T$ has P

Diagram:

$$\begin{tikzcd}
\text{scheme} & F \times_G T \arrow[r] \arrow[d, "f_T"'] \arrow[dr, phantom, "\square"] & F \arrow[d, "f"', "\text{rep.}"] & \text{sheaf} \\
\text{scheme} & T \arrow[r] \arrow[d, phantom, "\shortparallel" description] & G & \text{sheaf} \\
& h_T & &
\end{tikzcd}$$

Ex: If $f : X \to Y$ morphism of schemes, then $h_X \to h_Y$ repr. by schemes:

$$\begin{tikzcd}
h_{X \times_Y T} = h_X \times_{h_Y} h_T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & h_X \arrow[d] \\
h_T \arrow[r] & h_Y
\end{tikzcd}$$

If P stable property, then $h_X \to h_Y$ has P $\iff$ $X \to Y$ has P.
(immediate from defs, cf. Lemma 5.1.8)

§ Definition of algebraic space

Def (5.1.10) An algebraic space is a functor $X : Sch^{op} \longrightarrow Set$ (i.e. presheaf on Sch) such that:

(i) X is a sheaf in the étale topology

(ii) $\Delta_X : X \longrightarrow X \times X$ is represented by schemes

(iii) $\underbrace{\exists \text{ scheme } U \text{ and } U \longrightarrow X \text{ surjective étale}}_{= \text{ an étale presentation.}}$.

Lemma 5.1.9: Let $X \in Sh(Sch_E)$. TFAE

(1) Δ_X is represented by schemes.

(2) $\forall$ schemes T, T', $\forall\, T \xrightarrow{f} X$, $T' \xrightarrow{g} X$ the fiber product $T \times_X T'$ is a scheme

(3) $\forall$ scheme T, every map $T \longrightarrow X$ is represented by schemes.

proof: (2) $\Leftrightarrow$ (3) by defn.

(1) $\Rightarrow$ (2) follows from the cartesian square:

$$\begin{tikzcd} T \times_X T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \times T' \arrow[d, "f \times g"] \\ X \arrow[r, "\Delta_X"'] & X \times X \end{tikzcd}$$

(2) $\Rightarrow$ (1): Let $T \xrightarrow{(f,g)} X \times X$ be a map.

Then have cartesian squares:

$$\begin{tikzcd} T \times_{X \times X} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \arrow[d, "\Delta_T"] \arrow[dd, bend left=60, "(f,g)"] \\ T \times_{f,X,g} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T \times T \arrow[d, "f \times g"] \\ X \arrow[r] & X \times X \end{tikzcd}$$

a scheme by (2)

$\square$

In particular, in (iii), $U \longrightarrow X$ is represented by schemes so property surjective + étale makes sense.

§ Algebraic spaces as equivalence relations [O, 5.2]

<u>Def</u>: An étale equivalence relation of schemes is:

(i) an equivalence relation $R \overset{s}{\underset{t}{\rightrightarrows}} X$ of schemes,
i.e., a groupoid such that $R \xrightarrow[(s,t)]{} X \times X$ <u>monomorphism</u>

$\iff \forall T$: $R(T) \rightrightarrows X(T)$ equiv relation,
i.e., $R(T) \to X(T) \times X(T)$ injective.

(ii) s (or equiv t) is étale.

<u>Ex</u>: G discrete group $\curvearrowright X \Rightarrow G \times X \overset{\pi_2}{\underset{g}{\rightrightarrows}} X$ étale.

free action $\overset{\text{def}}{\iff}$ $G \times X \xrightarrow{(\pi_2, g)} X \times X$ monomorphism.
$(g, x) \longmapsto (x, gx)$

Given an étale eq rel $R \rightrightarrows X$ we obtain a quotient (= coequalizer) $X \to X/R$ in $Sh(Sch_E)$. An eff. epi.

<u>Rmk</u>: X/R is sheafification of presheaf $T \mapsto X(T)/R(T)$.

<u>Prop 5.2.5</u>: (i) X/R is an algebraic space
and $X \to X/R$ is an étale presentation.
(ii) If Y algebraic space and $X \to Y$ étale pres,
then Y quotient of $X \times_Y X \rightrightarrows X$.

pf: (ii) by definitions (note that $X \times_Y X$ is a scheme by representability of Δ_Y and that $X \to Y$ is an epi.)

(i) Let $Y = X/R$. By Exc 5.D, we have $R = X \times_Y X$. This gives the cartesian diagram:

$$\begin{tikzcd}
\overset{\text{scheme}}{R = X \times_Y X} \arrow[r, "\text{mono}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \overset{\text{scheme}}{X \times X} \arrow[d, "\text{epi}"] \\
Y \arrow[r, "\text{mono}", "\Delta_Y"'] & Y \times Y
\end{tikzcd}$$

Need to prove that Δ_Y representable.

When $R \to X \times X$ qcpt ($\Leftarrow$ Δ_Y qc if representable)
($\overset{}{\underset{\text{def}}{\Longleftrightarrow}}$ Y quasi-separated)

then $R \to X \times X$ quasi-finite + separated
$\Rightarrow$ quasi-affine.

In general, $R \to X \times X$ is only locally quasi-finite and separated.

Representability of Δ_Y now follows from lemma below. (axiom (ii) of alg. space).

$\Rightarrow$ $X \to Y$ representable. Diagram now implies that $X \to Y$ étale + surj (étale descent of étale) $\Rightarrow$ axiom (iii)

$$\begin{tikzcd}
R \arrow[r, "\text{étale + surj}"] \arrow[d] & X \arrow[d, "\text{epi}"] \\
X \arrow[r, "\text{repr.}"] & Y
\end{tikzcd}$$

<u>Lemma</u>: $f: V \longrightarrow W$ morphism of sheaves, $W' \longrightarrow W$ epi.
Let $f': V' \longrightarrow W'$ base-change.

(i) If f' is rep. by schemes & quasi-affine, so is f
(ii) ———————— " ———— loc. qfin + sep, so is f.

<u>proof</u>: (i) Let T scheme, $T \xrightarrow{g} W$. Since $W' \to W$ epi,
$\exists\, T' \xrightarrow{h} T$ surjective étale and

$$\begin{tikzcd} T' \arrow[r] \arrow[dr, "g"'] & W' \arrow[r] \arrow[d, phantom, "\circ"] & W \\ & T \arrow[ur, "h"'] & \end{tikzcd}$$

This gives:

(pull-back of f')

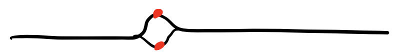
$$\begin{tikzcd}[column sep=large] \text{sch}\quad T' \times_{W'} V' \arrow[r, "q\text{-affine}"] \arrow[d] \arrow[dr, phantom, "\Box"] & T' \ \text{sch} \arrow[d, "h"', "\text{ét} + \text{surj}"] \arrow[r, bend left] & \\ T \times_W V \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & T \ \text{sch} \arrow[d, "g"'] \arrow[r, phantom, "\circ"] & W' \arrow[dl, "\text{epi}"] \\ V \arrow[r, "f"'] & W & \end{tikzcd}$$

By flat descent of quasi-affine morphisms
$\Longrightarrow T \times_W V \longrightarrow T$ is rep. and quasi-affine.

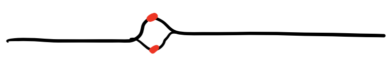
(ii) Reduces to (i) but a bit technical. $\Box$

§ Non-separated examples (created as quotients of free group actions)

(1) Consider non-sep affine line

$X = \mathbb{A}^1_{\mathbb{C}} \cup_{\mathbb{A}^1_{\mathbb{C}} \setminus 0} \mathbb{A}^1_{\mathbb{C}} =$ 

Let $\mathbb{Z}/2\mathbb{Z}$ act via switching the two copies/origins <u>and</u> by complex conjugation:

$X =$ 

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} \mathbb{C} \arrow[d] \\ X/\mathbb{Z}/2\mathbb{Z} \arrow[r] & \operatorname{Spec} \mathbb{R} \end{tikzcd}$$

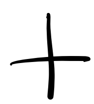
$\rightsquigarrow X/\mathbb{Z}/2\mathbb{Z} =$ 

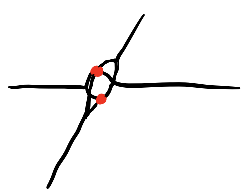
residue field is $\mathbb{C}$

$\mathbb{C}$

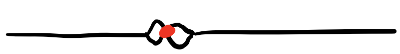
$(= \mathbb{A}^1_{\mathbb{R}}$ outside $0)$

$X/(\mathbb{Z}/2\mathbb{Z})$ is non-separated but "underlying" topological space looks like $\mathbb{A}^1_{\mathbb{R}}$. An algebraic space which is not a scheme.

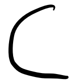
(2) Let $Y = \operatorname{Spec} k[x,y]/(xy) =$  and consider

$X = Y \cup_{Y \setminus 0} Y =$ 

Let $\mathbb{Z}/2\mathbb{Z}$ act by $x \longleftrightarrow y$ and switching the two copies of Y.

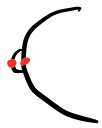
$X/\mathbb{Z}/2 =$  $(= \operatorname{Spec} k[xy]$ outside $0)$

1 point but infinitesimal nbhds look like 

(3) Let $Y = \operatorname{Spec} k[x]$ =  ramified 2:1 cover.

$\downarrow$

$S = \operatorname{Spec} k[t]$ = —— $(t = x^2)$

Let $X = Y \cup_{Y \setminus 0} Y$ =  and let $\mathbb{Z}/2\mathbb{Z}$ act by

$x \longmapsto -x$ + switch two copies / origins.

$\Rightarrow$ $X/(\mathbb{Z}/2\mathbb{Z})$ =  (= Spec $k[t]$ outside 0)

$\downarrow \pi$

S = ——

A "bug-eyed cover" of S: π birational and ramified at 0.

(4) Let $\underline{\mathbb{C}} = \coprod_{a \in \mathbb{C}} \operatorname{Spec} \mathbb{C}$ be the additive group $(\mathbb{C}, +)$ as a <u>discrete</u> group scheme (not quasi-compact)

$\underline{\mathbb{C}}$ acts on $\mathbb{A}^1_{\mathbb{C}}$ via $t \cdot z = z + t$.

There are 2 orbits in $|\mathbb{A}^1_{\mathbb{C}}|$:

(a) all closed points = all $\mathbb{C}$-points

(b) the generic point.

$\Rightarrow$ $\mathbb{A}^1_{\mathbb{C}} / \underline{\mathbb{C}}$ algebraic space with 2 points and <u>trivial</u> topology

(5) Let $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the discrete Galois group $\curvearrowright \operatorname{Spec} \overline{\mathbb{Q}}$.

Then $\operatorname{Spec} \overline{\mathbb{Q}} / \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ has 1 point but no residue field.

§ Separated examples (also quotients of schemes by free actions of finite grps)

Fact 1: Every separated noetherian 1-dim'l algebraic space is a (q-proj) scheme.

Fact 2: Every separated, ~~noeth.~~, regular 2-dim'l algebraic space is a (q-proj) scheme.
[of finite type / base scheme]
(unknown in general)

Ex: $\exists$ normal (toric) variety X of dim 2 which is proper but not projective.
Can equip X w/ free $\mathbb{Z}/2\mathbb{Z}$-action and $\exists$ orbit $\{x,y\}$ w/o common affine
$\Rightarrow$ $X/\mathbb{Z}/2\mathbb{Z}$ normal algebraic space of dim 2 which is proper
but not a scheme. (see Rmk below)

Ex (Hironaka) $\exists$ regular variety X of dim 3 which is proper but not projective

Can make this example so that $\mathbb{Z}/2\mathbb{Z} \curvearrowright X$ freely and exists
orbit $\{x,y\}$ w/o common affine nbhd

$\Rightarrow$ $X/\mathbb{Z}/2\mathbb{Z}$ regular alg. space of dim 3, proper but not scheme.
(See Rmk below)

(Read about Hironaka's 3-fold in Hartshorne, Appdx B
and Olsson, Ex. 5.3.2)

Rmk: G finite $\curvearrowright X$ freely
$\Rightarrow X \xrightarrow{\pi} X/G$ finite. If X/G scheme,
$\exists$ affine nbhd U of $\pi(x)$
$\Rightarrow \pi^{-1}(U)$ affine nbhd of orbit Gx

$$\begin{array}{ccccc}
Gx & \subset & X & \supset & \pi^{-1}(U) \\
\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow \\
\pi(x) & \in & X/G & \supset & U
\end{array}$$

§Proofs of facts 1+2:

Thm 1 (Chow lemma) X alg space.

(a) $\exists X' \to X$ finite surjective w/ X' scheme.

(b) (X sep, f.t./noeth) $\exists X' \to X$ proj birational w/ X' quasi-projective

Thm 2: Let X alg space and $X' \to X$ finite surjective.

(a) (Chevalley's thm) X' affine $\Rightarrow$ X affine

(b) (P. Gross '17) X' scheme w/ ample l.b. $\Rightarrow$ X scheme

(If in addition X normal, then X has an ample l.b.)

pf of Fact 1: By Thm 1(a) $\exists X' \to X$ finite surj w/ X' scheme.
For any $x \in X$, $X' \setminus p^{-1}(x)$ is affine $\Rightarrow$ $X \setminus x$ affine (Thm 2(a))

pf of Fact 2: By Thm 2(a), $\exists X' \to X$ proj birational w/ X' scheme.
Can prove (e.g. using flatification) that can assume X' sequence of blow-ups in smooth points $\Rightarrow$ X' smooth.
Then one proves that contracting $E \cong \mathbb{P}^1 \hookrightarrow X'$ with $E \cdot E = -1$ preserves quasi-projectivity.

Alternative proof if X proper: (following Goodman '69) Take $U \subset X$ affine open. Then $\exists$ eff. Cartier divisor $D \hookrightarrow X$ with $U = X \setminus D$ s.th. D satisfies Nakai's ampleness criterium. Using Thm 1(a) & 2(b) one sees that Nakai's criterion $\Rightarrow$ ample also for algebraic spaces.