Algebraic Stacks #5

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Stacks and torsors

§ Schemes are sheaves

Thm 4.1.2: Let $X \longrightarrow S$ morphism of schemes, then $h_X : Sch^{op}_{/S} \longrightarrow Set$ is a sheaf for étale/fppf/fpqc topologies.

Cor 4.2.12: Given $S' \longrightarrow S$ fppf covering, $\begin{matrix} X \\ \downarrow \\ S \end{matrix}$, $\begin{matrix} Y \\ \downarrow \\ S \end{matrix}$ schemes, then

$$Mor_S(X, Y) \longrightarrow Mor_{S'}(X', Y') \rightrightarrows Mor_{S''}(X'', Y'')$$

exact (where $X' = X \times_S S'$, $X'' \times_S S''$, $S'' = S' \times_S S'$).

proof: Sequence equals: $Y(X) \longrightarrow Y(X') \rightrightarrows Y(X'')$
and $X' \longrightarrow X$ fppf and $X'' = X' \times_X X'$. $\square$

(alt. proof (Olsson): Sheaves is a stack (uses fully faithfulness)))

§ $\mathcal{M}_S$ is a stack (corrected)

$$\mathcal{M}_g(S) \longrightarrow \mathcal{M}_g(S' \longrightarrow S) = \text{cat. of descent data.}$$

Fully faithful follows from 4.2.12:

$$Isom_S(X, Y) \longrightarrow Isom_{S'}(X', Y') \rightrightarrows Isom_{S''}(X'', Y'') \text{ equal.}$$

For ess. surj., see lecture #5.

§ Stacks vs sheaves

$F : \mathcal{D}^{op} \longrightarrow Cat$ pseudo-functor

Def (3.4.7)

- $\forall\, x, y \in F(T)$ define presheaves $\underline{Hom}(x,y)$, $\underline{Isom}(x,y)$

$\underline{Hom} : \mathcal{D}/_T^{op} \longrightarrow Set$ if $F : \mathcal{D}^{op} \to Grpd$

$T' \xrightarrow{f} T \longmapsto Hom(f^*x, f^*y) = Isom(f^*x, f^*y)$

- $\forall\, x \in F(T)$ define presheaf $\underline{Isom}(x)$ (of groups)

$$\mathcal{D}/_T^{op} \longrightarrow Grp$$
$$T' \xrightarrow{f} T \longmapsto Isom(f^*x)$$

Prop (4.6.2) $F(X) \xrightarrow{\varepsilon} F(\{U_i \to X\})$ fully faithful $\forall$ coverings $\{U_i \longrightarrow X\}$

$$\Updownarrow$$

$\underline{Hom}(x,y)$ sheaf $\forall\, x, y \in X$.

Def: F is called a <u>prestack</u> if the above equiv. conditions hold

	<u>Set-valued</u> functor		<u>Cat/Grpd</u>-valued functor
<u>Terminology</u>:	<u>presheaf</u> = functor	$\longleftrightarrow$	fibred category / pseudo-functor
	separated presheaf	$\longleftrightarrow$	<u>prestack</u> (bad terminology!)
	sheaf	$\longleftrightarrow$	stack

§ Stackification

pseudo-functor $F: \mathcal{D}^{op} \to Cat$

$\Updownarrow$

<u>Thm 4.6.5</u> $\mathcal{D}$ site. Given fibered category $F \to \mathcal{D}$, there exists a stack $F^a \to \mathcal{D}$ and morphism $F \to F^a$ such that

$$\mathrm{MOR}_{\mathcal{D}}(F^a, G) \longrightarrow \mathrm{MOR}_{\mathcal{D}}(F, G)$$

equivalence of categories for all stacks $G \to \mathcal{D}$. We call F^a the associated stack or stackification of F. If F CFG, then so is F^a.

<u>Rmk</u>: $(\ )^a$: FibCat $\rightleftarrows$ Stacks : forget, 2-adjunction.

§ Stack quotient of groupoids and group actions

<u>Ex 4.6.6</u>: Given groupoid of (pre)sheaves $R \rightrightarrows X$ on $\mathcal{D}$ can define fibered category $\{R \rightrightarrows X\}$ whose T-points are the groupoid $R(T) \rightrightarrows X(T)$. Even if R, X sheaves, $\{R \rightrightarrows X\}$ need not be a stack. We denote its <u>stackification</u> by $[R \rightrightarrows X]$.

<u>Ex</u>: G group scheme $\curvearrowright X$ scheme $\Rightarrow$ action groupoid $G \times X \rightrightarrows X$.

Let $[X/G] = [G \times X \rightrightarrows X]$.

<u>Ex</u>: $G \curvearrowright *$. Fibered category $\{*/G\} = \{G \rightrightarrows *\}$ has T-points $G(T) \rightrightarrows *$, i.e. one obj with auto $G(T)$.

Turns out that $BG := [*/G]$ has T-points $=$ <u>G-torsors</u> $/T$ which has much more objects. ($\{*/G\}(T) = \{\text{trivial } G\text{-torsor}\}$)

Rmk: If $R \rightrightarrows X$ equivalence relation, then $\{R \rightrightarrows X\}$ is fibered in set(oids) and $[R \rightrightarrows X]$ is usual sheafification.

Ex: Let $f : X \longrightarrow Y$ morphism of schemes. Gives equivalence relation $X \times_Y X \rightrightarrows X$. Even if f open covering

$$X = \coprod U_i \longrightarrow Y$$

the presheaf of sets $\{\coprod U_i \cap U_j \rightrightarrows \coprod U_i\}$ is not a sheaf (only separated): too few sections, e.g., $id_Y : Y \to Y$ does not lift unless one $U_i = Y$. Sheafification is Y.

In general, if f covering in the site, then $[X \times_Y X \rightrightarrows X] \xrightarrow{\cong} Y$.
($f : X \to Y$ is an effective epimorphism)

§ Torsors (Olsson 4.5)

$\mathcal{D}$ site, e.g., $\mathcal{D} = Sch/_S$ w/ étale / fppf (or fpqc) topology.

$G \to S$ group scheme or sheaf of groups over S.

$$\begin{array}{ccc} Sch/_S^{op} & \longrightarrow & Grp \\ T \to S & & G(T \to S) \end{array}$$

Def: A G-torsor is

- a sheaf $E : \mathcal{D}^{op} \longrightarrow Set$, with
- a (left) G-action on E

such that

(T1) $\exists$ a covering $\{S_i \to S\}$ s.th. $E(S_i) \neq \emptyset$ $\forall i$

(T2) whenever $E(T) \neq \emptyset$, then $G(T) \curvearrowright E(T)$ simply transitive.

Rmk:

- G-action on E means $G(T) \curvearrowright E(T)$ (group acting on set) functorially in $T \in \mathcal{D}$. Equivalent to map of sheaves $G \times E \xrightarrow{\rho} E$, $g, e \mapsto g.e$ satisfying axioms
  - $\rho \circ (id_G \times \rho) = \rho \circ (m \times id_E) : G \times G \times E \longrightarrow E$
  - $\rho \circ (e \times id_E) = id_E : E \longrightarrow G \times E \longrightarrow E$
- (T2) $\Leftrightarrow$ (T2'): $G \times E \xrightarrow[(\rho, \pi_2)]{} E \times E$ is an isomorphism
  $$g, e \longmapsto (g.e, e)$$
- If $E(T) \neq \emptyset$, then say that E is trivial over T.
  Given $p \in E(T)$, we obtain isomorphism $G \times T \longrightarrow E \times T$
  $$g \longmapsto g.p$$

SLOGAN: A G-torsor looks like G but w/o specified unit element,
= up to translation by G.

Morphism of G-torsors:

$$\begin{tikzcd} G\times E \arrow[r, "\mathrm{id}\times f"] \arrow[d, "\rho"'] \arrow[dr, phantom, "\circ"] & G\times E' \arrow[d, "\rho'"] \\ E \arrow[r, "f"] & E' \end{tikzcd}$$

Rmk: Automatically iso
locally $\exists p \in E(T)$
and then $f(p) \in E'(T)$
gives $\begin{tikzcd} G \arrow[r, "\cong", "p"'] \arrow[rr, bend right, "\cong\ f(p)"'] & E \arrow[r, "f"] & E' \end{tikzcd}$
so f locally iso $\Rightarrow$ f iso.

$\mathrm{Aut}(E) = G$.

§ Principal G-bundles

$D = Sch_{/S}$ w/ fppf topology.
$G \longrightarrow S$ flat group scheme, locally of finite presentation.

Def: A principal G-bundle is
- a scheme $P \longrightarrow S$, with
- a G-action $G\times P \xrightarrow{\rho} P$ $\quad (\rho\circ\rho = \rho\circ m,\ \rho\circ e = \mathrm{id})$

such that
(P1) $P \longrightarrow S$ is faithfully flat an loc. of f.p.
(P2) $G\times P \xrightarrow{(\rho,\pi_2)} P\times P$ is an isomorphism
$(g,p) \longmapsto (g\cdot p, p)$

Rmk: • Scheme w/ G-action $\rightsquigarrow$ sheaf h_P w/ h_G-action
• (P1) $\Rightarrow$ $\exists$ section fppf locally, e.g.

$$\begin{tikzcd} P\times_S P \arrow[r] \arrow[d] & P \arrow[d] \\ P \arrow[r, "\text{fppf cover}"'] \arrow[u, bend left, "\Delta"] & S \end{tikzcd}$$

$= (T1)$
• (P2) $\Leftrightarrow$ (T2)

So G-bundles $\longrightarrow$ G-torsors.

Ex (Galois theory) Let k field, G finite group.

__Trivial G-torsor:__

$$\begin{tikzcd} \coprod_{g\in G} \operatorname{Spec} k \arrow[d] \\ \operatorname{Spec} k \end{tikzcd}$$

$\circlearrowleft G$ permutes conn. components

__Galois extension:__

= connected G-torsor

$$\begin{tikzcd} \coprod_{\sigma\in G} \operatorname{Spec} K \arrow[r] \arrow[d, shift left] & \operatorname{Spec} K \arrow[d] \\ \operatorname{Spec} K \arrow[r] \arrow[u, shift left, bend left, "i_\sigma"] \arrow[ur, "\sigma"] & \operatorname{Spec} k \arrow[ul, phantom, "\lrcorner"] \end{tikzcd}$$

$\circlearrowleft G = \mathrm{Gal}(K/k) = \mathrm{Aut}_k(K)$

$i_\sigma \longleftrightarrow \sigma$

__General G-torsor:__

$$\begin{tikzcd} \coprod_{G} \operatorname{Spec} K \arrow[r] \arrow[d] & \coprod_{G/H} \operatorname{Spec} K \arrow[d] \\ \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

$H = \mathrm{Gal}(K/k)$

not nec normal

$\circlearrowleft G$ permutes conn. comp's via $G \curvearrowright G/H$ mult on left but also acts via auts of K.

Prop 4.5.6: If $G \to S$ is affine (and flat, l.f.p.) then

$$\{G\text{-bundles}/S\} \to \{G\text{-torsors}/S\}$$

equivalence of categories.

pf: • Fully faithful is clear (w/ natural definition of morphism of G-bdls)

• Essential surjectivity: Let E G-torsor $/S$. Then $\exists S' \to S$ fppf cover s.th. $E(S') \neq \emptyset$. Pick $p \in E(S')$. Then $G \times_S S' \to E \times_S S'$ isomorphism $\Rightarrow$ $E' := E \times_S S' \to S'$ affine + f. flat + l.f.p.

$$\begin{tikzcd} E'' \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\text{aff}"] & E' \arrow[r] \arrow[d, "\text{aff}"] & E \arrow[d] & \text{sheaf} \\ S'' \arrow[r, shift left] \arrow[r, shift right] & S' \arrow[r, "\substack{\text{fppf} \\ \text{cover}}"'] & S & \end{tikzcd}$$

By descent of affine, $E \to S$ is also affine.

—— " —— surj, flat, l.f.p., $E \to S$ is also such. $\square$

Rmk: If $G \to S$ is affine + smooth, then so is $P \to S$.

$\Rightarrow \exists$ étale surj $S' \to S$ s.th $P(S') \neq \emptyset$.

$$\Rightarrow \{\text{principal } G\text{-bundles}\} = \{G\text{-torsors in fppf top}\}$$
$$= \{G\text{-torsors in ét. top}\}$$

Ex: $\{\text{vector bundles}\} \overset{①}{\cong} \{GL_n\text{-torsors in Zar. top}\} \overset{②}{\cong} \{GL_n\text{-torsors in ét top}\}$

① $\mathcal{E} \longmapsto \underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}_S^{\oplus n})$ with GL_n-action $g.\varphi = g\varphi$

"ass. frame bundle"

$$\underline{\mathrm{Isom}}(\mathcal{E}, \mathcal{O}_S^{\oplus n})(T \to S) = \mathrm{Isom}(\mathcal{E}|_T, \mathcal{O}_T^{\oplus n})$$

② GL_n is "special" in the sense of Serre: this exactly means that Zar-torsor $\Leftrightarrow$ ét-torsor.

Rmk: When $G \to S$ flat but <u>not affine</u>, correct definition of principal G-bundle is to let $P \to S$ be an alg. space, then $\{\text{principal } G\text{-bundles}\} = \{G\text{-torsors}\}$.

Ex: There exists torsors under abelian varieties which are algebraic spaces but not schemes.

Ex 4.5.8 (μ_n-torsors via line bundles)

$\mu_n = \operatorname{Spec} \mathbb{Z}[t]/(t^n-1)$ group of n^{th} roots of unity

$\mu_n(S) = \{f \in \mathcal{O}_S : s^n = 1\}$

Kummer exact sequence:

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \xrightarrow{\wedge n} \mathbb{G}_m \longrightarrow 1$$
$$f \longmapsto f^n$$

exact seq of sheaves in fppf-topology:

$$\mathbb{G}_m(S) \longrightarrow \mathbb{G}_m(S)$$
$$f \longmapsto f^n$$

not surjective but $\exists\, n^{th}$ roots fppf-locally on S: take

$$S' = \operatorname{Spec}_S (\mathcal{O}_S[t]/t^n - f)$$

then t is n^{th} root of f:

$$\begin{tikzcd} \mathbb{G}_m(S) \arrow[r] \arrow[d] & \mathbb{G}_m(S) \ni f \arrow[d] \\ \mathbb{G}_m(S') \arrow[r] & \mathbb{G}_m(S') \end{tikzcd}$$
$$t \longmapsto f$$

If S has all n^{th} roots of unity ($\Rightarrow$ n invertible on S)
then $\mu_n \cong \mathbb{Z}/n\mathbb{Z}$.

Turns out that μ_n-torsors easier that $\mathbb{Z}/n\mathbb{Z}$-torsors due to Kummer sequence.

<u>Def:</u> Let Σ_n be following category: ("n-torsion l.b.'s")

$$\mathrm{Obj} = \{(\mathcal{L}, \sigma) : \mathcal{L} \text{ l.b. on } S, \ \sigma : \mathcal{L}^{\otimes n} \xrightarrow{\simeq} \mathcal{O}_S\}$$

$$\mathrm{Mor}((\mathcal{L},\sigma),(\mathcal{L}',\sigma')) = \left\{ \mathcal{L} \xrightarrow{f} \mathcal{L}' : \begin{tikzcd} \mathcal{L}^{\otimes n} \arrow[rr, "f^{\otimes n}"] \arrow[dr, "\sigma"'] & \circ & \mathcal{L}'^{\otimes n} \arrow[dl, "\sigma'"] \\ & \mathcal{O}_S & \end{tikzcd} \right\}$$

Functor $\Sigma_n \longrightarrow \mathrm{Tors}(\mu_n)$

$(\mathcal{L},\sigma) \longmapsto \underline{\mathrm{Isom}}((\mathcal{O}_S, \mathrm{id}_S), (\mathcal{L},\sigma))$

$$T \longmapsto \left\{ \mathcal{O}_T \xrightarrow[\simeq]{\lambda} \mathcal{L} : \begin{tikzcd} \mathcal{O}_T \arrow[rr, "\lambda^n"] \arrow[dr, "\mathrm{id}"'] & \circ & \mathcal{L}^{\otimes n} \arrow[dl, "\sigma|_T"] \\ & \mathcal{O}_T & \end{tikzcd} \right\}$$

Action: $\mu_n \curvearrowright \underline{\mathrm{Isom}}((\mathcal{O}_S, \mathrm{id}_S), (\mathcal{L},\sigma))$

on T-points $\quad \zeta \in \mu_n(T) = \{\zeta \in \mathcal{O}_T : \zeta^n = 1\}$

$\lambda \in \underline{\mathrm{Isom}}(T) = \{\lambda : \mathcal{O}_T \to \mathcal{L} : \sigma \circ \lambda^n = \mathrm{id}\}$

action is: $\zeta . \lambda = \zeta\lambda$.

This action is simply transitive:

- given λ_1, λ_2, then $\lambda_2 \lambda_1^{-1} = \zeta$ s.t. $\zeta^n = 1$.

and $\underline{\mathrm{Isom}}$ has sections fppf-locally: Zariski-locally $\mathcal{L} \cong \mathcal{O}$ and fppf-locally has $\sqrt[n]{\sigma}$. Take $\lambda = (\sqrt[n]{\sigma})^{-1}$.

If n invertible over S, $\mu_n \longrightarrow S$ is étale (locally $\cong \mathbb{Z}/n\mathbb{Z}$), and then μ_n-torsors have sections étale-locally.

Exc 4.G: $G \to S$ sheaf of groups. Prove that

$$BG : T \longmapsto \mathrm{Tors}(G \times_S T)$$

is a stack.

$$\begin{tikzcd} P' \arrow[r, "G\text{-equi}"] \arrow[d] \arrow[dr, phantom, "\square"] & P \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$