Algebraic Stacks #3

Feb 7, 2024

§ Fibered categories: motivation

Saw earlier that considering

$$M_g : Sch^{op} \longrightarrow Set$$

did not work (iso-trivial families). Thus want

$$\mathcal{M}_g : Sch^{op} \longrightarrow Grpd$$

$T \longmapsto$ groupoid: obj $C \longrightarrow T$ family of curves

morp.
$$\begin{tikzcd} C_1 \arrow[rr, "\cong"] \arrow[dr] & \circ & C_2 \arrow[dl] \\ & T & \end{tikzcd}$$

But somewhat awkward: given $T' \xrightarrow{g} T$, pullback

$$g^* : \mathcal{M}_g(T) \longrightarrow \mathcal{M}_g(T')$$

only defined up to isomorphism:
so involves a choice

$$\begin{tikzcd} \underset{= g^*(C)}{C \times_T T'} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ T' \arrow[r] & T \end{tikzcd}$$

$\rightsquigarrow$ if $T'' \xrightarrow{h} T' \xrightarrow{g} T$ composition, then functors $h^* \circ g^*$ and $(g \circ h)^*$ are only naturally isomorphic.

$\rightsquigarrow$ $\mathcal{M}_g : Sch^{op} \longrightarrow Grpd$ only <u>pseudo-functor</u>.

Classical solution: use <u>fibered categories</u> (Grothendieck 1959)

<u>Thm</u>: $\{F : Sch^{op} \longrightarrow Cat \quad \text{pseudo functor}\}$

$$\begin{tikzcd} F \arrow[d, mapsto] \\ \int F \end{tikzcd} \qquad \Vert\wr \quad \text{equivalence of 2-categories}$$

difficult to make precise. ($\exists$ maps in both directions and isomorphisms obj-wise etc)

$\{\mathcal{C} \xrightarrow{\pi} Sch^{op} \quad \text{fibered category}\}$

Construction of $\int F$ from F ("Grothendieck construction")

ob $\mathcal{C} = \{(T, x) : T \in Sch,\ x \in F(T)\}$

mor $\mathcal{C}$: $Mor((T_1, x_1), (T_2, x_2)) = \{(g, \varphi) : T_1 \xrightarrow{g} T_2,\ g^*(x_2) \xrightarrow{\varphi} x_1\}$

$\mathcal{C} \xrightarrow{\pi} Sch^{op}$, $\quad (T, x) \longmapsto T$
$\qquad\qquad\qquad\quad (g, \varphi) \longmapsto g$

<u>Ex</u>: M_g. ob $\mathcal{C} = \{ C \xrightarrow{f} T$ proper flat family of smooth conn curves of genus $g\}$

$$Mor(f_1, f_2) = \left\{ g : T_1 \to T_2 : \begin{tikzcd} C_1 \arrow[rr, "\cong", "\varphi"'] \arrow[dr] & & g^* C_2 \arrow[dl] \\ & T_1 & \end{tikzcd} \right\}$$

$$\overset{\text{can}}{=} \left\{ \begin{tikzcd} C_1 \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\square"] & C_2 \arrow[d] \\ T_1 \arrow[r, "g"'] & T_2 \end{tikzcd} \quad \text{cartesian diagram} \right\}$$

So can define $\mathcal{C}$ directly without any choices of $g^* C_2$ etc.

<u>But</u>, need to ensure that $g^* C_2$ exists.

§ Fibered categories: definition

<u>Def</u>: A **fibered category** is a functor $\pi : \mathcal{C} \longrightarrow \mathcal{D}$
such that $\forall (T' \xrightarrow{g} T) \in \mathcal{D}$ $\forall x \in \mathcal{C}(T)$
$\exists (x' \xrightarrow{\varphi} x) \in \mathcal{C}$ above g that is <u>cartesian</u> ← see below
↑ not nec unique!

<u>Notation</u>: Fiber $\mathcal{C}(T)$ = category above T

obj: $x \in \mathcal{C}$, $\pi(x) = T$

morp: $\varphi : x_1 \longrightarrow x_2$ s.th. $\pi(\varphi) = id_T$.

<u>Rmk</u>: $\mathcal{C} \longrightarrow \mathcal{D}$ Grothendieck construction to $F : \mathcal{D}^{op} \longrightarrow Cat$
gives $\mathcal{C}(T) = F(T)$.

<u>Def</u>:

$$\begin{tikzcd} x' \arrow[r, "\varphi"] \arrow[d, phantom, "\downarrow \pi"] & x & \in \mathcal{C} \\ T' \arrow[r, "g"'] & T & \in \mathcal{D} \end{tikzcd}$$

is **cartesian** if

$$\begin{tikzcd} \mathcal{C} \arrow[dd, "\pi"] & x'' \arrow[r, dashed, "\exists !"', "\lambda"] \arrow[rr, bend left, "\psi"] & x' \arrow[r, "\varphi"'] & x \\ & & & \\ \mathcal{D} & T'' \arrow[r, "f"'] \arrow[rr, bend left, "gf"] & T' \arrow[r, "g"'] & T \end{tikzcd}$$

$\forall f : T'' \longrightarrow T'$

$\forall \psi : x'' \longrightarrow x$ above gf

$\exists ! \lambda : x'' \longrightarrow x'$ above f

s.th. $\varphi \circ \lambda = \psi$

$x' \longrightarrow x$ is analogue in fibered category
of pull-back $g^*(x)$ in functor $\mathcal{D}^{op} \longrightarrow Cat$.

x' is called **a pull-back** of x along g.

If $x'' \to x$, $x' \to x$ are pull-backs of x along $g : T' \to T$
then $\exists! \, x'' \to x'$, $x' \to x''$ above $\mathrm{id}_{T'}$ $\Longrightarrow$ $x' \cong x'' \in \mathcal{C}(T')$.

$$\begin{tikzcd}
x' \arrow[r] \arrow[d, shift left, dashed, "\exists!"] & x \\
x'' \arrow[ur] \arrow[u, shift left, dashed, "\exists!"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
T' \arrow[r] \arrow[d, equal] & T \\
T' \arrow[ur] &
\end{tikzcd}$$

__Slogan__: pull-backs unique up to unique isomorphism.

__Ex__: $\mathcal{M}_g$.

$$\begin{tikzcd}
 & C \arrow[d] \\
T' \arrow[r, "g"] & T
\end{tikzcd}
\Rightarrow \exists
\begin{tikzcd}
\boxed{C \times_T T'} \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
T' \arrow[r, "g"] & T
\end{tikzcd}$$

not unique, only unique up to unique iso

(φ, g) __cartesian__:

$$\begin{tikzcd}
C'' \arrow[r, dashed, "\exists!"] \arrow[rr, bend left] \arrow[d] \arrow[dr, phantom, "\square"] & C \times_T T' \arrow[r, "\varphi"] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\
T'' \arrow[r] & T' \arrow[r, "g"] & T
\end{tikzcd}$$

By __def__, big square
$\begin{tikzcd} C'' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & C \arrow[d] \\ T'' \arrow[r] & T \end{tikzcd}$
is cartesian.

Since right square is cartesian, $\exists! \, C'' \to C \times_T T'$.
This also makes left square cartesian.

§ Morphisms of fibered categories

Fibered categories $/\mathcal{D}$ is a 2-category:

$\mathrm{MOR}\left(\mathcal{C}_1 \xrightarrow{\pi_1} \mathcal{D},\ \mathcal{C}_2 \xrightarrow{\pi_2} \mathcal{D}\right)$ is a 1-category

obj: $f : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ functor, strictly commuting: $\pi_2 \circ f = \pi_1$

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "f"] \arrow[dr, "\pi_1"'] & & \mathcal{C}_2 \arrow[dl, "\pi_2"] \\ & \mathcal{D} & \end{tikzcd}$$

s.th. F(cartesian) is cartesian.

morp: $f_1 \xrightarrow{\tau} f_2$ base preserving natural thm of functors

$$\begin{tikzcd} x_1 \arrow[r, "\varphi"] \arrow[d, phantom, "\downarrow \pi_1"] & x_2 \\ T_1 \arrow[r, "g"'] & T_2 \end{tikzcd} \qquad \begin{tikzcd} f_1(x_1) \arrow[r, "f_1(\varphi)"] \arrow[d, "\tau(x_1)"'] & f_1(x_2) \arrow[d, "\tau(x_2)"] \\ f_2(x_1) \arrow[r, "f_2(\varphi)"] & f_2(x_2) \end{tikzcd} \xrightarrow{\pi_2} \begin{tikzcd} T_1 \arrow[r, "g"] \arrow[d, "id"'] & T_2 \arrow[d, "id"] \\ T_1 \arrow[r, "g"] & T_2 \end{tikzcd}$$

(i.e., $\forall x \in \mathcal{C}_1(T)$, $\tau(x): f_1(x) \longrightarrow f_2(x)$ morphism in $\mathcal{C}_2(T)$)

Ex:

$$\begin{tikzcd} \mathcal{M}_{g,n} \arrow[rr, "f"] \arrow[dr] & & \mathcal{M}_g \arrow[dl] \\ & Sch & \end{tikzcd} \qquad \left\{ \begin{tikzcd} C \arrow[d] \\ T \arrow[u, bend left, "s_1"] \arrow[u, bend right, "s_n"'] \end{tikzcd} \right\} \longrightarrow \left\{ \begin{tikzcd} C \arrow[d] \\ T \end{tikzcd} \right\}$$

$$\begin{tikzcd} \mathcal{M}_g \arrow[rr, hook, "\text{fully faithful}"] \arrow[dr] & & \overline{\mathcal{M}}_g \arrow[dl] \\ & Sch & \end{tikzcd} \qquad \left\{ \begin{tikzcd} C \arrow[d] \\ T \end{tikzcd} \text{ smooth} \right\} \longrightarrow \left\{ \begin{tikzcd} C \arrow[d] \\ T \end{tikzcd} \text{ poss. nodal} \right\}$$

Def:
$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "f"] \arrow[dr] & \circ & \mathcal{C}_2 \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$
equiv if $\exists$
$$\begin{tikzcd} \mathcal{C}_2 \arrow[rr, "g"] \arrow[dr] & \circ & \mathcal{C}_1 \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$
$g \circ f \cong \mathrm{id}_{\mathcal{C}_1}$
$f \circ g \cong \mathrm{id}_{\mathcal{C}_2}$

base-preserving

Lem 3.1.8 + Prop 3.1.10
$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "f"] \arrow[dr] & \circ & \mathcal{C}_2 \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$
fully faithful $\Leftrightarrow$ $f(T)$ f.f. $\forall T \in \mathcal{D}$.
equiv $\Leftrightarrow$ equiv

Analogue: $\mathcal{F} \longrightarrow \mathcal{G}$ morphism of presheaves
inj/bij $\Leftrightarrow$ $\mathcal{F}(U) \longrightarrow \mathcal{G}(U)$ inj/bij $\forall U$.

§ Categories fibered in sets

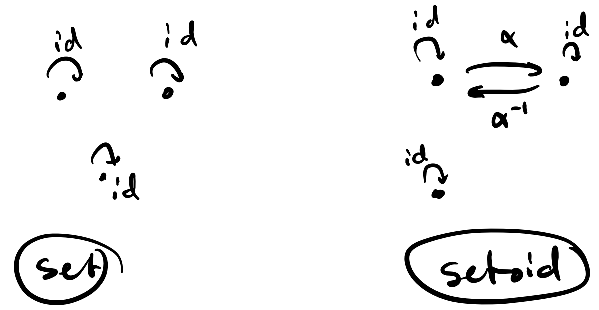
Def: $\mathcal{C} \xrightarrow{\pi} \mathcal{D}$ **fibered in sets** if $\mathcal{C}(T)$ "is" a set $\forall T \in \mathcal{D}$

i.e. a category w/ only identity morphisms.

$\mathcal{C} \xrightarrow{\pi} \mathcal{D}$ **fibered in setoids** if $\mathcal{C}(T)$ is equiv. to a set $\forall T \in \mathcal{D}$

i.e. a category where all morphisms invertible and only auto is id.

Picture:



Prop 3.2.8: Equivalences:

$$\{\text{categories fibered in sets}\} \longleftrightarrow \left\{\begin{array}{l}\text{presheaves on } \mathcal{D} \\ \text{i.e. functors } \mathcal{D} \to \mathrm{Set}\end{array}\right\}$$

$$\updownarrow$$

$$\{\text{categories fibered in setoids}\}$$

pf: From setoids to sets: pick "skeleton" $\mathcal{C}' \subset \mathcal{C}$

i.e. pick one $x \in \mathcal{C}'(T)$ for each equiv class. Then
$$\begin{tikzcd} \mathcal{C}' \arrow[rr] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$
equiv.

From CFS to presheaves: for $x \in \mathcal{C}(T)$, $g: T' \to T$

$\exists$ unique cartesian map $x' \to x$ above g. Let $g^*(x) = x'$.

This defines a functor $D^{op} \to \mathrm{Set}$. $\square$

§ 2-Yoneda lemma

Recall: $\mathcal{D} \longrightarrow PShv(\mathcal{D}, Set)$ presheaf of sets.

$T \longmapsto h_T = Mor(-, T)$

$PShv(\mathcal{D}, set) \cong CFS(\mathcal{D})$

$$\begin{tikzcd} h_T \arrow[r, mapsto] & \mathcal{D}/T \arrow[d, "\pi"] \\ & \mathcal{D} \end{tikzcd} = \{ X \to T \} \qquad \begin{tikzcd} X \to T \arrow[d, mapsto, "\pi"] \\ X \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[d, "g"'] \arrow[r] & T \\ X_2 \arrow[ur] & \end{tikzcd} \ \mapsto \ \begin{tikzcd} X_1 \arrow[d, "g"'] \\ X_2 \end{tikzcd}$$

__Prop 3.2.2__ (2-Yoneda lemma)

$$MOR(\mathcal{D}/T, \mathcal{C}) \longrightarrow \mathcal{C}(T)$$
$$f \longmapsto f(id_T)$$

is an equivalence of categories.

__Cor__: $\mathcal{D} \to CFS$ is fully faithful

$$T \longmapsto \begin{tikzcd} \mathcal{D}/T \arrow[d] \\ \mathcal{D} \end{tikzcd}$$

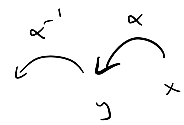
pf: Apply Prop to $\mathcal{C} = \mathcal{D}/S \Rightarrow Mor(\mathcal{D}/T, \mathcal{D}/S) = \mathcal{D}/S(T) = \{T \to S\}$

§ Groupoids

Def 1: A groupoid is a (small) category s.th all morphisms are invertible.

Def 2: A groupoid is

- a set of objects X,
- a set of relations/arrows R,
- maps
  $s : R \longrightarrow X$ (source)
  $t : R \longrightarrow X$ (target)
  $e : X \longrightarrow R$ (unity)
  $\imath : R \longrightarrow R$ (inverse)
  $c : R \times_{s,X,t} R \longrightarrow R$ (composition)



such that all the natural axioms holds:

- $s \circ e = t \circ e = id_X$
- $s \circ \imath = t, \quad t \circ \imath = s$
- $s \circ c = s \circ \pi_2, \quad t \circ c = t \circ \pi_1$

$\Big\}$ "basic axioms"

- $c \circ (id_R \times e) = c \circ (e \times id_R) = id_R$ — unit axiom
- $c \circ (\imath, id_R) = e \circ s, \quad c \circ (id_R, \imath) = e \circ t$ — inverse axiom
- ...

Pictured as $R \overset{s}{\underset{t}{\rightrightarrows}} X$ although full data is

$$\begin{tikzcd} R \times_{s,X,t} R \arrow[r, "c"] & R \arrow[loop above, "\imath"] \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & X \arrow[l, "e"'] \end{tikzcd}$$

Ex:

$$\begin{tikzcd} x \arrow[loop above, "id"] \arrow[r, bend left, "f"] & y \arrow[loop above, "id"] \arrow[l, bend left, "f^{-1}"] \end{tikzcd}$$

def 1

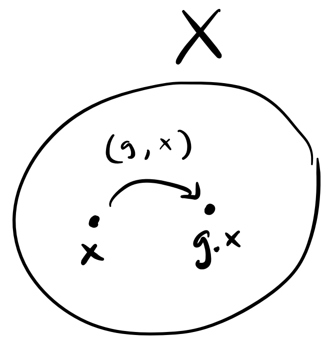
$$\{id_x, f, f^{-1}, id_y\} \overset{s}{\underset{t}{\rightrightarrows}} \{x, y\}$$

def 2

<u>Ex</u> (action groupoid) G group acting on set X (on the left).

$$\sigma : G \times X \longrightarrow X$$
$$g, x \longmapsto g.x$$

Action groupoid $G \times X \underset{t = \sigma}{\overset{s = \pi_2}{\rightrightarrows}} X$. Picture



TFAE:

(1) The action is free (i.e., $\mathrm{stab}(x) = \{e\}$ $\forall x \in X$)

(2) The action groupoid $G \times X \rightrightarrows X$ is a setoid.

(3) $(s, t) : G \times X \longrightarrow X \times X$ is injective.
$$(g, x) \longmapsto (x, g.x)$$

Then the groupoid is equivalent to the set X/G.

<u>Ex</u> (equivalence relation) $\sim$ equivalence relation on X

$R = \{(x, y) : x \sim y\} \subset X \times X$

$R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X$ is naturally a groupoid. This groupoid is equivalent to the set $X/\sim$.

<u>Rmk</u>: $(s, t) : R \longrightarrow X \times X$ is injective.

<u>Ex</u>: Given $X \xrightarrow{f} Y$ we get equivalence relation $x_1 \sim x_2$ if $f(x_1) = f(x_2)$. Equivalently described as $R = X \times_Y X \subset X \times X$.

Rmk: Let $\tau = (R \rightrightarrows X)$ groupoid. TFAE

(1) $\mathrm{stab}(x) = s^{-1}(x) \cap t^{-1}(x) = \{ \mathrm{id}_x \}$ $\forall x \in X$.

(2) τ is a setoid.

(3) $$(s,t) : R \longrightarrow X \times X \quad \text{is injective}$$
$$r \longmapsto (s(r), t(r))$$

(4) $R \rightrightarrows X$ arises from an equivalence relation on X.

§ Categories fibered in groupoids

<u>Def</u>: $\mathcal{C} \xrightarrow{\pi} \mathcal{D}$ category fibered in groupoids (CFG)
if fibered category s.th. $\mathcal{C}(X)$ <u>groupoid</u> $\forall X \in \mathcal{D}$,

<u>Prop</u>: $\mathcal{C}$ CFG $\Leftrightarrow$ all arrows are cartesian.

<u>pf</u>: Let $x \xrightarrow{\varphi} y \in \mathcal{C}$ above $\pi(x) \xrightarrow{g} \pi(y) \in \mathcal{D}$

Since fibered category, $\exists$ cartesian arrow $x' \to y$ above g and

$$\begin{tikzcd} x' \arrow[r, "\text{cart}"] \arrow[d, dashed, "\exists!\ \alpha"'] & y \\ x \arrow[ur, "\varphi"'] & \end{tikzcd} \quad \text{above} \quad \begin{tikzcd} \pi(x) \arrow[r] \arrow[d, equal, "id"'] & \pi(y) \\ \pi(x) \arrow[ur] & \end{tikzcd}$$

If $\mathcal{C}$ CFG then α invertible $\Rightarrow$ φ also cartesian.

Conversely, if all arrows cartesian, let α arrow in $\mathcal{C}(X)$, i.e.
$x' \xrightarrow{\alpha} x \in \mathcal{C}$ above $\underbrace{\pi(x') = \pi(x)}_{= X} \in \mathcal{D}$

Then α cartesian implies α invertible. $\square$

§ Fiber products of groupoids

Groupoids constitute a 2-category (like Cat).
(Actually a (2,1)-category which simplifies things.)

Recall fiber product of sets:

$$\begin{tikzcd} X \times_S Y \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

As set $X \times_S Y = \{ (x,y) : f(x) = g(y) \}$

Univ property: $\forall$

$$\begin{tikzcd} T \arrow[rrd, bend left, "v"] \arrow[rdd, bend right, "u"'] & & \\ & \arrow[r, phantom, "\circ"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

can uniquely be filled:

$$\begin{tikzcd} T \arrow[rrd, bend left, "v"] \arrow[rdd, bend right, "u"'] \arrow[rd, dashed, "\exists ! \alpha"] & & \\ & X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"'] & S \end{tikzcd}$$

Notation: $\alpha = (u, v)$

Equality of objects in groupoids bad:

Ex: $\mathcal{C} = \begin{tikzcd} a \arrow[loop above, "id"] \arrow[loop below, "\tau"] \end{tikzcd}$ $\quad \mathcal{D} = \begin{tikzcd} x \arrow[loop above] \arrow[loop below] \arrow[r, shift left] & y \arrow[l, shift left] \arrow[loop above] \arrow[loop below] \end{tikzcd}$ equivalent groupoids.

$\mathcal{C} \xrightarrow{F} \mathcal{D}$, $a \longmapsto x$ and $\mathcal{C} \xrightarrow{G} \mathcal{D}$, $a \longmapsto y$ are naturally isomorphic $F \Rightarrow G$

$$\begin{tikzcd} & \mathcal{C} \arrow[d, "G"', "\cong"] \\ \mathcal{C} \arrow[r, "\cong", "F"'] & \mathcal{D} \end{tikzcd}$$

Fiber product $\mathcal{C} \times_{\mathcal{D}} \mathcal{C}$ should be equivalent to $\mathcal{C} \times_{\mathcal{C}} \mathcal{C} \cong \mathcal{C}$ but $F(a) \neq G(a)$.

Def: Given $\mathcal{C}_1 \xrightarrow{f} \mathcal{D}$ and $\mathcal{C}_2 \xrightarrow{g} \mathcal{D}$ morphism of groupoids (= functors of categories), the **2-fiber product** $\mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2$ is the groupoid with

objects: $\{(x, \varphi, y) : x \in \mathcal{C}_1, y \in \mathcal{C}_2, f(x) \xrightarrow{\varphi} g(y)\}$

arrows: $\mathrm{Map}\big((x,\varphi,y), (x',\varphi',y')\big)$
$= \{(\alpha, \beta) : \varphi' \circ f(\alpha) = g(\beta) \circ \varphi\}$

automatically invertible since arrow in groupoid

$$x \xrightarrow{\alpha} x' \qquad y \xrightarrow{\beta} y'$$

$$\begin{tikzcd} f(x) \arrow[r, "f(\alpha)"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\circ"] & f(x') \arrow[d, "\varphi'"] \\ g(y) \arrow[r, "g(\beta)"'] & g(y') \end{tikzcd}$$

Gives

$$\begin{tikzcd} \mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\tau \nearrow"] & \mathcal{C}_2 \arrow[d, "g"] \\ \mathcal{C}_1 \arrow[r, "f"'] & \mathcal{D} \end{tikzcd}$$

where $\tau : f \circ \pi_1 \Rightarrow g \circ \pi_2$ is the natural isomorphism defined by $\tau(x, \varphi, y) = \varphi$.

__Universal__ __property__: given 2-commutative diagram:

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, bend left, "v"] \arrow[dr, bend right, "u"'] & & \mathcal{C}_2 \arrow[d, "g"] \\
& \mathcal{C}_1 \arrow[r, "f"'] \arrow[ur, phantom, "\psi \Nearrow"] & \mathcal{D}
\end{tikzcd}
$$

there exists

$$
\begin{tikzcd}
\mathcal{C} \arrow[dr, red, "\alpha"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] \arrow[dr, phantom, shift left=6, red, "\beta_2 \Nearrow"] \arrow[dr, phantom, shift right=6, red, "\beta_1 \Nearrow"'] & & \\
& \mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\tau \Nearrow"] & \mathcal{C}_2 \arrow[d, "g"] \\
& \mathcal{C}_1 \arrow[r, "f"'] & \mathcal{D}
\end{tikzcd}
$$

such that δ, γ, τ glues to ψ.

Moreover, the data $(\alpha, \beta_1, \beta_2)$ is unique up to unique 2-isomorphism, i.e., given

$$
\begin{tikzcd}
\mathcal{C} \arrow[dr, red, "\tilde{\alpha}"] \arrow[drr, bend left, "v"] \arrow[ddr, bend right, "u"'] \arrow[dr, phantom, shift left=6, red, "\tilde{\beta}_2 \Nearrow"] \arrow[dr, phantom, shift right=6, red, "\tilde{\beta}_1 \Nearrow"'] & & \\
& \mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\tau \Nearrow"] & \mathcal{C}_2 \arrow[d, "g"] \\
& \mathcal{C}_1 \arrow[r, "f"'] & \mathcal{D}
\end{tikzcd}
$$

There $\exists!\, \gamma : \alpha \Rightarrow \tilde{\alpha}$ s.th. $\tilde{\beta}_2 \circ \gamma = \beta_2$, $\gamma \circ \beta_1 = \tilde{\beta}_1$

$$
\begin{tikzcd}
\mathcal{C} \arrow[dr, bend left, "\tilde{\alpha}"] \arrow[dr, bend right, "\alpha"'] \arrow[dr, phantom, "\Rightarrow\gamma"] \arrow[rrd, bend left, "v"] \arrow[ddr, bend right, "u"'] \arrow[rd, phantom, shift left=6ex, "\Rightarrow\tilde{\beta}_2"] \arrow[dd, phantom, shift left=4ex, "\Rightarrow\beta_1"] & & \\
& \mathcal{C}_1 \times_{\mathcal{D}} \mathcal{C}_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & \mathcal{C}_2 \\
& \mathcal{C}_1 &
\end{tikzcd}
$$