Algebraic stacks #2

Jan 31, 2024

§ Étale maps

§ Why étale topology? (analytical vs formal vs étale)

§ Grothendieck topologies

§ Group schemes as functors

§ Étale maps

Def: f étale if it is

(1) locally of finite presentation, and

(2) formally étale, i.e., $\forall$

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r] \arrow[d, hook, "\text{nilpotent immersion}"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A' \arrow[r] \arrow[ur, dashed, "\exists !"] & Y \end{tikzcd}$$

$A = A'/I$, $I^N = 0$

Rmk: formally smooth $=$ as above but $\exists^{\geq 1}$

formally unramified $=$ as above but $\exists^{\leq 1}$

Ex: $A = k$, $A' = k[\varepsilon]/(\varepsilon^2)$. Then formally étale/smooth/unramified says $T_x X \to T_{f(x)} Y$ bijective/surjective/injective.

Prop: TFAE

(1) f étale

(2) f smooth and rel dim 0

(3) f flat, l.f.p., and unramified $\Leftrightarrow$ fibers are étale

Ex: $A \longrightarrow B = A[t]_g/(f)$ $\quad f, g \in A[t]$, f monic.

Étale if $\frac{\partial f}{\partial t}$ invertible in B. Called "standard étale"

Ex: $k[x,y] \longrightarrow k[x,y,t]_{y-2t^2}/(t^3 - xt + y)$ $\quad f' = 3t^2 - x$

standard étale $\quad (f') = (y - 2t^2) = (1)$

Ex: $A \longrightarrow B = A[t]/(f)$ $\quad f = x^n + \cdots$ $\quad (f, f') = (1)$

$\Rightarrow$ finite étale.

Ex: $\operatorname{Spec} k' \longrightarrow \operatorname{Spec} k$ étale $\Leftrightarrow$ k'/k finite separable.

Def: $(X, x) \longrightarrow (Y, y)$ <u>étale nbhd</u> if $f: X \longrightarrow Y$ étale, $f(x) = y$.
Nisnevich nbhd if in addition $\kappa(y) \xrightarrow{\cong} \kappa(x)$.
(= strict ét nbhd = strongly ét nbhd = elementary ét nbhd

henselization $\quad$ strict henselization

Def: X scheme: $x \in |X|$

$$\mathcal{O}_{X,x} \longrightarrow \mathcal{O}^h_{X,x} \longrightarrow \mathcal{O}^{sh}_{X,x}$$

$$\mathcal{O}_{X,x} = \varinjlim_{\substack{U \subset X \\ x \in U \text{ open nbhd}}} \mathcal{O}_{U,x}, \qquad \mathcal{O}^h_{X,x} = \varinjlim_{\substack{(X',x') \to (X,x) \\ \text{Nis nbhd}}} \mathcal{O}_{X',x'}, \qquad \mathcal{O}^{sh}_{X,x} = \varinjlim_{\text{étale nbhd}} \mathcal{O}_{X',x'}$$

<u>Precise definition of</u> $\mathcal{O}^{sh}_{X,x}$: Need $\operatorname{Spec} k \xrightarrow{\bar{x}} X$ with k <u>separably closed</u>

$$\mathcal{O}^{sh}_{X,\bar{x}} = \varinjlim_{\substack{X' \xrightarrow{\text{étale}} X \\ \bar{x}' \nwarrow \;\; \nearrow \bar{x} \\ \operatorname{Spec} k}} \mathcal{O}_{X',\bar{x}'}$$

Ex: $X = \operatorname{Spec} k$, $\mathcal{O}^{sh}_{X,x} = k^{sep}$.

§ Why étale topology?

X scheme of finite type / $\mathbb{C}$

$\rightsquigarrow X^{an} = X(\mathbb{C})$ w/ Euclidean topology (+ structure as analytic space)

① Implicit function theorem

$$\begin{tikzcd} f: X \arrow[r] & Y \\ x \arrow[u, phantom, "\in" rotate=90] \arrow[r, mapsto] & y \arrow[u, phantom, "\in" rotate=90] \end{tikzcd}$$

f smooth at x,

- $f^{an}: X^{an} \to Y^{an}$ locally admits section $s: Y^{an} \to X^{an}$ with $s(y) = x$.

$\Rightarrow$ $\Updownarrow$

- $\exists$

$$\begin{tikzcd} & x \in U \arrow[dl, hook'] \arrow[dr, hook] & \\ X^{an} \arrow[dr] & & Y^{an} \times \mathbb{C}^n \arrow[dl] \\ & Y & \end{tikzcd}$$

$\Rightarrow \hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{Y,y}[[t_1, \dots, t_n]]$

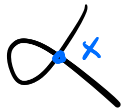
Étale version:

$$\exists \quad \begin{tikzcd} & U \arrow[dl, hook', "\text{Zariski open}"'] \arrow[dr, "\text{étale}"] & \\ X \arrow[dr] & \circ & Y \times \mathbb{A}^n \arrow[dl] \\ & Y & \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} & x & X \arrow[d] \\ y' & Y' \arrow[ur, "s"] \arrow[r, "\text{étale}"] & Y \\ y' \arrow[uur, mapsto, bend left] \arrow[rr, mapsto] & & y \end{tikzcd}$$

$\exists$ section, étale-locally on Y.

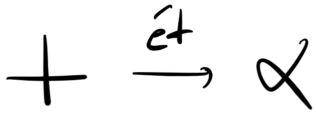
② Branches

$X =$  irreducible

X^{an} not irreducible, locally at x.

$\hat{\mathcal{O}}_{X,x} \cong k[[x,y]]/(xy)$ not domain.

$\exists\, X' \longrightarrow X$ étale s.th. X' connected, not irreducible.



③ Fundamental group

$X' \longrightarrow X^{an}$ covering space $\Rightarrow$ $\exists$ can. analytic structure on X'
(so that $X' \longrightarrow X^{an}$ finite, loc iso)

If X projective, then $\exists$ can scheme structure on X' such that
$X' \longrightarrow X^{an}$ finite étale.

Ex: $\mathbb{A}^1 \setminus 0 \longrightarrow \mathbb{A}^1 \setminus 0$ finite étale of deg n.
$z \longmapsto z^n$

$k[t] \longrightarrow k[t][x]/(x^n - t)$ $\quad f = x^n - t \quad f' = n x^{n-1}$

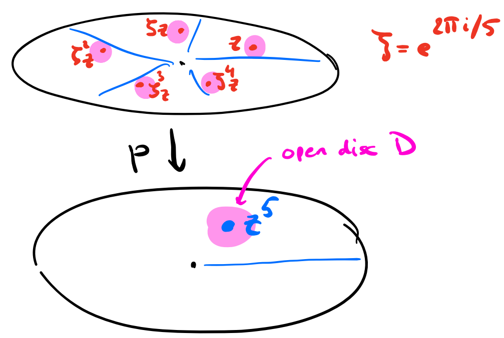
étale outside $t = 0$ $\quad (f, f') = (t)$

④ Group actions / torsors

G finite group $\curvearrowright X$ Hausdorff.

If action free $\Rightarrow$ $X \longrightarrow X/G$ principal G-bundle

$= G$-torsor



Ex:
$$\begin{array}{ccc} X = \mathbb{A}^1_{\mathbb{C}} \setminus 0 & \qquad & z \\ \downarrow p & & \big\downarrow \\ Y = \mathbb{A}^1_{\mathbb{C}} \setminus 0 & & z^5 \end{array}$$

$p^{-1}(y) = \{z, \zeta z, \zeta^2 z, \zeta^3 z, \zeta^4 z\}$ where $z^5 = y$. Covering space.

$\mathbb{Z}/5\mathbb{Z} \curvearrowright X \qquad z \longmapsto \zeta z \qquad$ Free action. $X/_{\mathbb{Z}/5\mathbb{Z}} \cong Y$.

Not locally trivial for Zariski top: $\forall U \overset{\text{open}}{\subset} \mathbb{A}^1 \setminus 0$, $p^{-1}(U)$ irreducible.
$X(\mathbb{C}) \to Y(\mathbb{C})$ loc. trivial for Eucl. top: $\forall D \subset \mathbb{A}^1 \setminus 0$ open disc
$p^{-1}(D) \cong D \amalg \cdots \amalg D$.

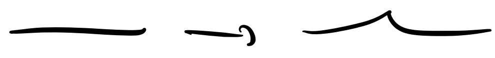
Upshot: p loc. trivial for étale topology.

<u>Facts:</u> $f: X \to Y$ morphism of schemes finite type $/\mathbb{C}$

f étale $\iff f^{an}$ local isomorphism (of analytical spaces)

$\implies f^{an}$ local homeomorphism (of topological spaces)

Not quite $\Leftarrow$

<u>Ex:</u>  homeo, but not étale.

$$\operatorname{Spec} k[t] \qquad \operatorname{Spec} k[x,y]/(y^2-x^3)$$

$$t^2 \leftarrow\!\shortmid x$$
$$t^3 \leftarrow\!\shortmid y$$

f finite étale $\implies f^{an}$ covering space

§ Grothendieck topologies

X top. space. $\mathcal{F}$ presheaf on X.

$\iff$ functor $\mathrm{Op}(X)^{op} \longrightarrow \mathrm{Set}$

$\mathrm{Op}(X) = \{U \subset X \text{ open}\} = \{U \longrightarrow X \text{ open embedding}\}$

$\mathcal{F}$ sheaf if $\forall U \subset X$, $\forall U = \bigcup U_i$

$$\mathcal{F}(U) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

Definition 2.1.2. Let C be a category. A *Grothendieck topology on C* consists of a set $\mathrm{Cov}(X)$ of collections of morphisms $\{X_i \to X\}_{i \in I}$ for every object $X \in C$ such that the following hold:

(i) If $V \to X$ is an isomorphism then $\{V \to X\} \in \mathrm{Cov}(X)$.
(ii) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$ and $Y \to X$ is any arrow in C, then the fiber products $X_i \times_X Y$ exist in C and the collection

$$\{X_i \times_X Y \to Y\}_{i \in I}$$

is in $\mathrm{Cov}(Y)$.
(iii) If $\{X_i \to X\}_{i \in I} \in \mathrm{Cov}(X)$, and if for every $i \in I$ we are given $\{V_{ij} \to X_i\}_{j \in J_i} \in \mathrm{Cov}(X_i)$ then the collection of composites

$$\{V_{ij} \to X_i \to X\}_{i \in I, j \in J_i}$$

is in $\mathrm{Cov}(X)$.

Remark 2.1.3. We call the collections $\{X_i \to X\}_{i \in I}$ in $\mathrm{Cov}(X)$ the *coverings of X*.

Remark 2.1.4. The above notion of Grothendieck topology is called a 'pre-topology' in [8].

Definition 2.1.5. A category with a Grothendieck topology is called a *site*.

Examples of sites

(a) small Zariski site on a fixed scheme X

$$\mathcal{C} = \mathrm{Op}(X), \quad \mathrm{Cov}(U) = \{(U_i \subset U)_{i \in I} : \coprod U_i \to U \text{ surj}\}$$

(b) big Zariski site on all schemes.

$$\mathcal{C} = \mathrm{Sch}, \quad \mathrm{Cov}(X) = \{(U_i \subset X)_{i \in I} : \coprod U_i \longrightarrow X \text{ surj}\}$$

or Aff, $\mathrm{Sch}^{\mathrm{sep}}$, $\mathrm{Sch}_{\mathrm{f.t.}/S}$, $\mathrm{Sch}_{/S}$, ...

(c) small étale site on a scheme X

$$\mathcal{C} = \mathrm{\acute{E}t}(X) = \{U \longrightarrow X \text{ étale}\}$$

$$\mathrm{Cov}(U) = \{(U_i \xrightarrow{\text{ét}} U)_{i \in I} : \coprod U_i \longrightarrow U \text{ surjective}\}$$

(d) big étale site

$$\mathcal{C} = \mathrm{Sch} \quad \mathrm{Cov}(X) = \{(U_i \xrightarrow{\text{ét}} X)_{i \in I} : \coprod U_i \longrightarrow X \text{ surj}\}$$

(e) fppf site (always big) (fppf = fidèlement plat et de présentation finie)

$$\mathcal{C} = \mathrm{Sch} \quad \mathrm{Cov}(X) = \{(U_i \underset{\text{flat and loc. of fin. pres.}}{\longrightarrow} X)_{i \in I} : \coprod U_i \longrightarrow X \text{ surj}\}$$

Rmk: open immersion $\Rightarrow$ étale $\Rightarrow$ flat + l.f.p. $\Rightarrow$ open

(i.e. $f(\text{open})$ is open)

Consequence: If X quasi-compact, $(U_i \to X)_{i \in I}$ covering in any of the topologies above, then $\exists$ finite subcovering $(U_i \to X)_{i \in J}$

$J \subset I$ finite

Topological invariance of étale site

Thm [0, 1.3.14] $X' \to X$ univ homeo of schemes

$\Rightarrow \text{Ét}(X) \xrightarrow{\simeq} \text{Ét}(X')$ equiv of categories

§ Sheaves

A presheaf (= functor) $\mathcal{F} : \mathcal{C}^{op} \longrightarrow \mathcal{D}$ is a <u>sheaf</u> for the topology τ (or on the site $(\mathcal{C}, \tau)$) if $\forall X \in \mathcal{C}$, $\forall (U_i \to X)_{i \in I} \in \mathrm{Cov}^{\tau}(X)$

$$\mathcal{F}(X) \longrightarrow \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_X U_j)$$

equalizer.

<u>Thm</u> [0, 2.2.4] The fully faithful functor

$$\mathrm{Shv}^{\tau}_{\mathcal{D}}(\mathcal{C}) \xrightarrow{\quad \iota \quad} \mathrm{PShv}_{\mathcal{D}}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{D})$$

has a left adjoint $(-)^{\sim}$ called <u>sheafification</u>.

$\mathcal{F} \longrightarrow \iota(\tilde{\mathcal{F}})$ unit of above adjunction.

$\mathcal{C}$ category
$(\mathcal{C}, \tau)$ site
$\mathrm{Shv}^{\tau}(\mathcal{C})$ topos = category of sheaves of sets on a site.

§ Group schemes

A group scheme G over a scheme S is

① A scheme $G \to S$ together with
- multiplication $m: G \times_S G \to G$
- unit $e: S \to G$
- inverse $i: G \to G$

satisfying usual axioms

$\Updownarrow$

② A functor $G: Sch_{/S} \to Grp$ such that composition $Sch_{/S}^{op} \to Grp \to Set$ is a scheme.

<u>Ex</u>: $\operatorname{Spec} A \longmapsto A^\times$ is a functor $Aff^{op} \to Grp$

$\Rightarrow$ scheme $\mathbb{G}_m$ is a group scheme.

Given group scheme $G: Sch_{/S} \to Grp$ in ② obtain maps in ① via:

$$m(T \xrightarrow{x} S) : (G \times_S G)(T \xrightarrow{x} S) \longrightarrow G(T \xrightarrow{x} S)$$
$$\| $$
$$G(T \xrightarrow{x} S) \times G(T \xrightarrow{x} S)$$

given by multiplication in group $G(T \to S)$.

etc.