Algebraic stacks #15

May 20, 2024

Deligne–Mumford stacks as ringed toposes (Lecture by Errol)

Def: Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topos. We say that it is Deligne–Mumford if $\exists$ objects U_α of $\mathcal{X}$ such that:

(a) $\coprod_\alpha U_\alpha \longrightarrow 1$ is an epimorphism

(b) $\forall \alpha$: the ringed topos $(\mathcal{X}/_{U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha})$ is equivalent to $\mathrm{Sp\acute{e}t}(R_\alpha)$ for some ring R_α.

Def: A ringed topos is a pair $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ where $\mathcal{X}$ is a topos $(= \mathrm{Sh}(\mathcal{C}, J))$ and $\mathcal{O}_{\mathcal{X}}$ is a ring object in $\mathcal{X}$ $(= \mathcal{C}^{op} \longrightarrow \mathrm{Ring}$ which is a J-sheaf$)$

A map of ringed toposes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \longrightarrow (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a pair (f, α) where $f: \mathcal{X} \longrightarrow \mathcal{Y}$ is a geometric morphism and

$$\alpha: \mathcal{O}_{\mathcal{Y}} \longrightarrow f_* \mathcal{O}_{\mathcal{X}} \text{ map of ring objects in } \mathcal{Y}.$$

Can also define 2-cells... $\rightsquigarrow$ 2-category RingTopos.

Ex: If $(X, \mathcal{O}_X)$ ringed space, then $(\mathrm{Sh}(X), \mathcal{O}_X)$ ringed topos.

Ex: If $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \in \mathrm{RingTopos}$, $U \in \mathcal{X}$, then $(\mathcal{X}/_U, \mathcal{O}_{\mathcal{X}}|_U) \in \mathrm{RingTopos}$.

Def: Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a ringed topos, consider $\mathcal{O}_{\mathcal{X}}^\times$ $\left(\text{given by } \begin{array}{ccc} \mathcal{O}_{\mathcal{X}}^\times & \longrightarrow & \mathcal{O}_{\mathcal{X}} \times \mathcal{O}_{\mathcal{X}} \\ \downarrow & \lrcorner & \downarrow m \\ 1 & \xrightarrow{1} & \mathcal{O}_{\mathcal{X}} \end{array}\right)$

If

(1) "$0 \neq 1$"

$$\begin{array}{ccc} \emptyset & \longrightarrow & \mathcal{O}_{\mathcal{X}}^\times \\ \downarrow & \lrcorner & \downarrow \\ 1 & \xrightarrow{0} & \mathcal{O}_{\mathcal{X}} \end{array}$$

(2) "$\forall s$: $s \in \mathcal{O}_{\mathcal{X}}^{\times}$ or $1-s \in \mathcal{O}_{\mathcal{X}}^{\times}$"

$$\mathcal{O}_{\mathcal{X}}^{\times} \amalg \mathcal{O}_{\mathcal{X}}^{\times} \xrightarrow{(e,\, 1-e)} \mathcal{O}_{\mathcal{X}}$$ is an epi where $\mathcal{O}_{\mathcal{X}}^{\times} \xhookrightarrow{e} \mathcal{O}_{\mathcal{X}}$

Then we say that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is <u>locally ringed topos</u>

<u>Rmk</u>: If $\mathcal{X}$ has enough points, then enough that the stalks are local rings.

<u>Def</u>: A ringed topos $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is <u>strictly henselian</u> if for every ring R and finite collection of étale maps $R \to R_\alpha$ s.th. $R \to \prod_\alpha R_\alpha$ is faithfully flat, then the map $\coprod_\alpha \mathcal{O}_{\mathcal{X}}^{R_\alpha} \to \mathcal{O}_{\mathcal{X}}^{R}$ is an epimorphism where $\mathcal{O}_{\mathcal{X}}^{R}$ is the object of $\mathcal{X}$ defined by:

$$\mathcal{X}(U, \mathcal{O}_{\mathcal{X}}^{R}) \cong \mathrm{Ring}(R, \mathcal{X}(U, \mathcal{O}_{\mathcal{X}})) \qquad (\text{naturally in } U \in \mathcal{X})$$

<u>Rmk</u>: The usual $\operatorname{Spec} R$ is locally ringed but not strictly henselian.

<u>Def</u>: (Étale topos of a ring R) Consider the category of étale R-algebras $\mathrm{CAlg}_R^{\text{ét}}$ with the topology on $(\mathrm{CAlg}_R^{\text{ét}})^{op}$ generated by finite family of maps $\{A \to A_i\}$ s.th. $A \to \prod_\alpha A_\alpha$ f. flat. We call $\mathrm{Sh}(\mathrm{CAlg}_R^{\text{ét}})$ the étale topos of R.

<u>Def</u>: $\mathcal{O}_R : \mathrm{CAlg}_R^{\text{ét}} \to \mathrm{Set} : (R \to A) \longmapsto A$.

<u>Prop</u>: $\mathcal{O}_R$ is a sheaf of rings for the above topology which is strictly henselian.

<u>Def</u>: $\operatorname{Spét}(R) = (\mathrm{Sh}(\mathrm{CAlg}_R^{\text{ét}}), \mathcal{O}_R)$ is the étale spectrum of R.

Prop: $\underbrace{\mathrm{sHenTopos}((\mathcal{X},\mathcal{O}_{\mathcal{X}}),\ \mathrm{Sp\acute{e}t}\,R)}_{\text{a priori just a category}} \simeq \underbrace{\mathrm{Ring}(R,\Gamma(\mathcal{X},\mathcal{O}_{\mathcal{X}}))}_{\text{a set (in particular a groupoid)}}$

Cor: $\mathrm{sHenTopos}(X,Y)$ groupoid $\forall X,Y$ DM-stacks.

Rmk: If $X=(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ is Deligne–Mumford, then it is strictly henselian.
(one shows that strictly henselian cover $\Rightarrow$ strictly henselian)

Def: Given $X=(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ DM-stack, we get a pseudo-functor:

$$h_X : \mathrm{Ring} \longrightarrow \mathrm{Grpd} : R \longmapsto \mathrm{sHenTopos}(\mathrm{Sp\acute{e}t}\,R,\ X)$$

Thm: This defines a fully faithful functor

$$\text{DM-ringed toposes} \longrightarrow \mathrm{Grpd}^{\mathrm{Ring}}$$

and its image is the $(2,1)$-category of DM-stacks.

proof (idea): If $X=(\mathcal{X},\mathcal{O}_{\mathcal{X}})$ and $Y=(\mathcal{Y},\mathcal{O}_{\mathcal{Y}})$ DM, then $\forall U\in\mathcal{X}$ consider

$$\Theta_U : \mathrm{DM}(X/U,\ Y) \longrightarrow \mathrm{Grpd}^{\mathrm{Ring}}(h_{X|_U},\ h_Y)$$

Call U "good" object if Θ_U is an equivalence.

(i) $U\longmapsto\Theta_U$ takes coproducts to products (so good obj's closed under small coproducts)

(ii) Given epi $U_0\longrightarrow A$ in $\mathcal{X}$ get $\cdots\ \underset{U_0\times_A U_0\times_A U_0}{\underset{\shortparallel}{U_2}} \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \underset{U_0\times_A U_0}{\underset{\shortparallel}{U_1}} \rightrightarrows U_0 \longrightarrow A$ and if U_i good $\forall i$, then A good.

(iii) Every affine U is good (Yoneda)
(iv) If $f: A \hookrightarrow B$ mono and B good, then A good.
(v) If $f: A \to B$ arb. and B good, then A good.
(vi) $\forall A$, $\exists$ cover $U_0 := \coprod_\alpha U_\alpha \to A$, U_α affine.
By (iii) + (v) $U_{\alpha_1} \times_A \cdots \times_A U_{\alpha_n}$ good. By (i) $U_0, U_1, \ldots$ good.
By (ii) A is good.

Lurie's geometries (Lecture by Gabriel) (cf. Lurie DAG V, §1-2)

Comparisons: Spec / Spét
scheme / DM-stacks
⋮

Def: If $\mathcal{C}$ category, $\mathcal{X}$ topos, then $\mathrm{Shv}_{\mathcal{C}}(\mathcal{X}) = \mathrm{Continuous}(\mathcal{X}^{op}, \mathcal{C})$
$= \mathrm{Right\ adj}(\mathcal{X}^{op}, \mathcal{C})$

Ex: $\mathcal{C} = \mathcal{S}$ cat of sets. $\mathrm{Shv}_{\mathcal{S}}(\mathcal{X}) \simeq$ adj $\mathcal{X}^{op} \rightleftarrows \mathcal{S}$ (T) $\simeq \mathcal{X}$

Ex: $\mathcal{C} = \mathrm{CRing}$. $\mathrm{Shv}_{\mathrm{CRing}}(\mathcal{X})$ = internal rings in $\mathcal{X}$.

More generally, for any category $\mathcal{G}$:

$$\mathrm{Shv}_{\mathrm{Lex}(\mathcal{G}, \mathrm{Set})}(\mathcal{X}) = \mathrm{Right\ adj}(\mathcal{X}^{op}, \mathrm{Lex}(\mathcal{G}, \mathrm{Set}))$$
$$= \mathrm{Left\ adj}(\mathrm{Lex}(\mathcal{G}, \mathrm{Set})^{op}, \mathcal{X})$$
$$= \mathrm{Lex}(\mathcal{G}, \mathcal{X})$$

e.g., $\mathcal{G} = \mathrm{CRing}_{\omega}^{op}$ gives

$$\mathrm{Shv}_{\mathrm{CRing}}(\mathcal{X}) \simeq \mathrm{Lex}(\mathrm{CRing}_{\omega}^{op}, \mathcal{X})$$

Idea (for Lurie geometries) Abstract the structure on $\mathrm{CRing}_\omega^{op}$ needed to def. an internal locally ringed object.

Def: A geometry $\mathcal{G}$ is a small lex category,
- the data of admissible maps: a subcategory $\mathcal{G}^{ad} \subseteq \mathcal{G}$ closed under id, comp and pull-backs and
$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "g"'] & & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$
if $g, h \in \mathcal{G}^{ad} \Rightarrow f \in \mathcal{G}^{ad}$.
- a Grothendieck topology on $\mathcal{G}$ s.th. each covering contains a covering generated by adm. maps.

$\boxed{\text{Ex: } \mathcal{G} = \mathrm{CRing}_\omega^{op} \quad A \to A[\tfrac{1}{a}] \text{ adm}}$

Ex: $\left(\mathrm{Sp}(A[\tfrac{1}{a_\alpha}]) \to \mathrm{Sp}(A)\right)_\alpha$ covering if $(a_\alpha) = (1)$ in A.

Def: A $\mathcal{G}$-structure on $\mathfrak{X}$ is a functor $\mathcal{O} : \mathcal{G} \xrightarrow{lex} \mathfrak{X}$ s.th. for any covering of adm maps $\{U_\alpha \to X\}$, the
$$\coprod \mathcal{O}(U_\alpha) \longrightarrow \mathcal{O}(X) \quad \text{is an epi in } \mathfrak{X}$$

Def: We have a category $\mathrm{Struc}_{\mathcal{G}}(\mathfrak{X})$ given by $\mathcal{G}$-str + nat. trans.
and subcategory $\mathrm{Struc}_{\mathcal{G}}^{loc}(\mathfrak{X})$ given by $\mathcal{G}$-str + nat. trans such that $\forall\, U \to X$ adm, the naturality square is a pull-back

Ex: $\mathcal{G}_{Zar} = \mathrm{CRing}_\omega^{op}$ w/ $A \to A[\tfrac{1}{a}]$ as admissible

Ex: $\mathfrak{X} = \mathrm{Sets}$. Then a $\mathcal{G}_{Zar}$-structure on $\mathfrak{X}$ is a local ring R b/c

$\mathrm{CRing} \simeq \mathrm{Lex}(\mathrm{CRing}_\omega^{op}, \mathfrak{X})$
$R \longmapsto (A \mapsto \mathrm{Ring}(A, R))$

and $\coprod_\alpha \mathrm{Ring}(A[\tfrac{1}{a_\alpha}], R) \to \mathrm{Ring}(A, R)$ surj
for $A = \mathbb{Z}[x]$, $(1) = (x, 1-x)$ implies
$\forall r \in R$, either r or $1-r$ invertible.

If $R \xrightarrow{\varphi} S$ ring homo, then map in $\mathrm{Struc}^{loc}_{\mathcal{G}}(\mathrm{Sets}) \iff \varphi$ local b/c

$$\begin{array}{ccc} R^\times = \mathrm{Hom}(\mathbb{Z}[x][\tfrac{1}{x}], R) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}[x][\tfrac{1}{x}], S) = S^\times \\ \downarrow & & \downarrow \\ R = \mathrm{Hom}(\mathbb{Z}[x], R) & \longrightarrow & \mathrm{Hom}(\mathbb{Z}[x], S) = S \end{array}$$

so pull-back square $\iff \varphi$ local.

Rmk: If $\mathcal{X}$ has enough points, we can test the locality of a $\mathcal{G}$-map on stalks

Ex: If $\mathcal{G}$ is lex, there is a discrete geometry $\mathcal{G}_{disc}$ where adm $\iff$ iso and the topology is trivial. In this case $\mathrm{Struc}_{\mathcal{G}_{disc}}(\mathcal{X}) \cong \mathrm{Shv}_{\mathcal{G}}(\mathcal{X})$

Prop: $\exists$ a classifying topos for a geometry, that is, $\mathrm{Struc}_{\mathcal{G}}(\mathcal{X}) \simeq \mathrm{Topos}(\mathcal{X}, \mathcal{K})$ but this is not enough to recover $\mathrm{Struc}^{loc}_{\mathcal{G}}(\mathcal{X})$

$$\mathcal{G} \xrightarrow{\text{Yoneda}} \hat{\mathcal{G}} \xrightarrow{\text{Sheafify}} \mathrm{Shv}(\mathcal{G}) = \mathcal{K}$$

(This doesn't use the admissible maps?)

How to define a $\mathcal{G}$-scheme

In classical theory, $(X, \mathcal{O}_X)$ locally ringed space is a scheme if locally of the form $(\mathrm{Sp}\,A, \mathcal{O}_{\mathrm{Sp}\,A})$.

Affine scheme: $(\mathrm{Spec}\,A, \mathcal{O}_{\mathrm{Spec}\,A})$ has univ. property:

Spec A is the loc ringed space $(X, \mathcal{O}_X)$ univ w/ a map $A \to \Gamma(X, \mathcal{O}_X)$

$$\mathrm{Hom}_{LRS}((X,\mathcal{O}_X), (\mathrm{Spec}\,A, \mathcal{O}_{\mathrm{Spec}\,A})) \xrightarrow{\cong} \mathrm{Hom}_{RS}((X,\mathcal{O}_X), (*, \mathbb{R}))$$
$$\cong \mathrm{Hom}_{Ring}(A, \Gamma(X, \mathcal{O}_X))$$

$$\begin{array}{ccc} \mathrm{LocRingSp} & \overset{\top}{\hookrightarrow} & \mathrm{RingSp} \\ \| & & \| \\ \mathrm{Top}(\mathcal{G}_{Zar}) & \xrightarrow{\top} & \mathrm{Top}(\mathcal{G}_{disc}) \end{array}$$

$\mathrm{Top}(\mathcal{G})$ = topos w/ a $\mathcal{G}$-structure

We have a map $\mathcal{G}_{disc} \longrightarrow \mathcal{G}_{Zar}$. We ask for a right adjoint of $\mathrm{Top}(\mathcal{G}_{Zar}) \to \mathrm{Top}(\mathcal{G}_{disc})$

In general, for a map of geometries $\mathcal{G} \longrightarrow \mathcal{G}'$ have a functor $\mathrm{Top}(\mathcal{G}') \longrightarrow \mathrm{Top}(\mathcal{G})$.

<u>Fact</u>: This map has a right adjoint $\mathrm{Spec}^{\mathcal{G}'}_{\mathcal{G}}$.

In particular:

$$\begin{tikzcd} \mathrm{Str}_{\mathcal{G}_{disc}}(\mathrm{Set}) \arrow[r] \arrow[rr, bend right, "\mathrm{Spec}_{\mathcal{G}}"'] & \mathrm{Top}(\mathcal{G}_{disc}) \arrow[r, "\mathrm{Spec}^{\mathcal{G}_{disc}}_{\mathcal{G}}"] & \mathrm{Top}(\mathcal{G}) \end{tikzcd}$$

This generalizes usual spectrum

<u>Def</u>: A $\mathcal{G}$-<u>affine scheme</u> is a $\mathcal{G}$-structure of the form $\mathrm{Spec}_{\mathcal{G}}(X)$ for some $X \in \mathrm{Struc}_{\mathcal{G}_{disc}}(\mathrm{Set})$

A $\mathcal{G}$-<u>scheme</u> is a $\mathcal{G}$-structure s.th $\exists \{U_\alpha\}$ in $\mathcal{X}$ s.th $\coprod U_\alpha \to \mathcal{X}$ epi and $\forall \alpha$: $(\mathcal{X}|_{U_\alpha}, \mathcal{O}_{\mathcal{X}}|_{U_\alpha})$ affine $\mathcal{G}$-scheme.

$\mathcal{G}_{et}(k) = \mathrm{CRing}_{k,\omega}^{op}$, $\quad$ f admissible if f étale.

$\{A \to A_\alpha\}$ covering if $A \to \prod A_\alpha$ faithfully flat.

$\mathcal{G}_{Zar}(k) = \mathrm{CRing}_{k,\omega}^{op}$, $\quad$ f admissible if $A \to A[\frac{1}{a}]$

$\{A \to A[\frac{1}{a_\alpha}]\}_\alpha$ covering if $(a_\alpha) = (1)$

$\mathcal{O} : \mathcal{G}_{et}(k) \xrightarrow{\text{lex}} \mathrm{Set}$ correspond to $A \in k\text{-alg}(\mathrm{Set})$

$\mathcal{O}$ is a $\mathcal{G}_{et}(k)$-structure $\iff$ $\forall B$ k-alg and $\{B \xrightarrow{et} B_\alpha\}$ s.th. $B \to \prod B_\alpha$ f.flat

$$\underbrace{\coprod \mathcal{O}(B_\alpha)}_{\mathrm{Ring}(B_\alpha, A)} \longrightarrow \mathcal{O}(B) \quad \text{surjective}$$

$\iff$ A strictly henselian