Algebraic stacks #14
May 8, 2024

§ Analytic and topological stacks (Lecture by Benedetta)

Thm: Every simply connected Riemann surface is conformally equivalent to one of
- $\mathbb{H}$ hyperbolic
- $\mathbb{C}$ Euclidean
- $\mathbb{S}^1$ $(=\mathbb{P}^1_{\mathbb{C}})$ spherical

Thm (unif for stacks) Every simply connected DM analytic curve is equiv to
- $\mathbb{H}$ hyperbolic
- $\mathbb{C}$ Euclidean
- $\mathcal{P}(m,n)$

Note: everything is finite type, separated, smooth in this talk.

Def: A map $f: \mathcal{Y} \longrightarrow \mathcal{X}$ of topological stacks $(\mathrm{Top}^{op} \longrightarrow \mathrm{Grpd}$ + stack cond$)$
is representable if $\forall X \longrightarrow \mathcal{X}$, $X \in \mathrm{Top}$, $Y := X \times_{\mathcal{X}} \mathcal{Y}$ is in Top.
If P property of maps stable under base change, f has this property P
if $\forall X \to \mathcal{X}$, $Y \to X$ has P.
$\mathcal{X}$ is a pre-DM top stacks if $\exists p: X \longrightarrow \mathcal{X}$ epi and rep. by local homeo
$\mathcal{X}$ is a DM top stack if pre-DM s.th. $\Delta_{\mathcal{X}}$ is repr. by closed map
which has finite discrete fibers.

Analogue for (smooth) analytic stacks replacing Top with Comp = complex manifolds.

Rmk: $\mathrm{AlgDM}/_{\mathbb{C}} \longrightarrow \mathrm{AnDM} \longrightarrow \mathrm{TopDM}$

$$\pi_n(\mathcal{X}, x) := [(\mathbb{S}^n, \cdot), (\mathcal{X}, x)]$$

If $\mathcal{X}$ locally path connected and semilocally 1-connected, then

subgroups of $\pi_1(\mathcal{X}, x)$ $\longleftrightarrow$ covering spaces $\mathcal{Y} \to \mathcal{X}$ up to equivalence

$\mathcal{X}$ <u>uniformizable</u> if universal cover is in Top.

<u>Fact</u>: $\exists\, \mathcal{X} \xrightarrow{\text{nat.}} \mathcal{X}_{mod} = |\mathcal{X}|$

TopDM $\longrightarrow$ Top

AnDM $\longrightarrow$ An

<u>Ex</u>: • $G \curvearrowright X$ $|[X/G]| = X/G$.

• $\mathbb{Z}/n\mathbb{Z} \curvearrowright \mathbb{D}$ $\mathcal{X} = [\mathbb{D}/\mathbb{Z}_n]$ one orbifold point $B\mathbb{Z}/n\mathbb{Z}$ at origin

• $\mathbb{Z}/n\mathbb{Z} \curvearrowright \mathbb{S}^2$ $\mathcal{X} = [\mathbb{S}^2/\mathbb{Z}_n]$ two —— " —————— at north & south pole

rotation by $2\pi/n$

• $\mathcal{P}(m,n)$ weighted projective line $= [\mathbb{C}^2 \setminus \{0\} / \mathbb{C}^\times]$, $m,n \geq 1$

$t \in \mathbb{C}^\times$ $t.(x,y) = (t^m x, t^n y)$

– $|\mathcal{P}(m,n)| = \mathbb{P}^1$, $\mathcal{P}(1,1) = \mathbb{P}^1$

– If $(m,n)=1$, then $\mathcal{P}(m,n)$ orbifold w/ 2 orbifold pts $[1:0]$, $[0:1]$ w/ stabilizers $\mathbb{Z}/m\mathbb{Z}$, $\mathbb{Z}/n\mathbb{Z}$

– If $(m,n)=d$, then $\mathcal{P}(m,n)$ $\mathbb{Z}/d\mathbb{Z}$-gerbe over $\mathcal{P}(m/d, n/d)$

• $\mathcal{F}(m,n)$ football of type (m,n) = orbifold w/ underlying space $\mathbb{P}^1$ and 2 orbifold pts $B\mathbb{Z}/m$ and $B\mathbb{Z}/n$. (Can be assumed to be $0, \infty$.)

Then $\mathcal{F}(m,n) \cong \mathcal{P}(m,n)$ if $(m,n)=1$.

Prop: Every DM top stack is *locally a quotient stack*, that is:

$\exists\ \mathcal{X} = \bigcup \mathcal{U}_i$ open cov. s.th. $\mathcal{U}_i = [X_i / G_i]$.

Prop: $\mathcal{X}$ top DM. For $x \in |\mathcal{X}|$ have grp homo $\varphi_x : I_x \longrightarrow \pi_1(\mathcal{X}, x)$.

(i) If $\mathcal{Y} \longrightarrow \mathcal{X}$ cover, then $\mathcal{Y} \in \mathrm{Top}$ iff $\forall x \in \mathcal{X}$ φ_x is injective and the conjugate of the subgroup H of $\pi_1(\mathcal{X}, x)$ corresponding to $\mathcal{Y}$ don't intersect

(ii) $\mathcal{X}$ uniformizable iff φ_x injective $\forall x \in \mathcal{X}$.

Prop: If $\mathcal{U} \in \mathrm{AnDM}$ orbifold, $|\mathcal{U}| = \mathbb{D}$ (or $\mathbb{C}$) and $\exists\ 1$ orbifold pt. Then $\mathcal{U} = [\mathbb{D} / \mathbb{Z}/n\mathbb{Z}]$ (or $[\mathbb{C}/\mathbb{Z}/n\mathbb{Z}]$) for some n.

Thm: $\mathcal{X} \in \mathrm{AnDM}$. Then $\mathcal{X}$ is an H-gerbe over an orbifold $\mathcal{Y}$. If $\mathcal{X}' \to \mathcal{X}$ map and both H-gerbes, then

$$\begin{tikzcd} \mathcal{X}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\circ"] & \mathcal{X} \arrow[d] \\ \mathcal{Y}' \arrow[r] & \mathcal{Y} \end{tikzcd}$$

§ Uniformization for stacks (Lecture by Jochen)

First: Uniformization for orbifold curves

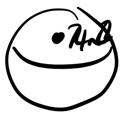
New (equiv) def of fundamental group:

Orbifold curve $\mathcal{C}$ = Riemann surface w/ holes $Q_1, \ldots, Q_\ell$ and orbifold pts $P_1, \ldots, P_k$
genus g of order $n_1, \ldots, n_k$

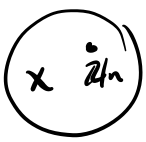
To compute $\pi_1(\mathcal{C})$: Let $U = \mathcal{C} \setminus \{P_1, \ldots, P_k\}$.

$$\pi_1(U) = \mathbb{Z}^{*\, 2g+k+\ell-1} = \mathbb{Z}a_i * \mathbb{Z}b_i * \mathbb{Z}\rho_j * \mathbb{Z}\sigma_h \Big/ \left\langle \prod [a_i, b_i] \prod \rho_j \prod \sigma_h = 1 \right\rangle$$

$$\pi_1(\mathcal{C}) = \pi_1(U) \Big/ \left\langle \text{extra relations } \rho_j^{n_j} = 1 \right\rangle$$

Ex: •  $\mathcal{F}(1,n) = \mathcal{P}(1,n)$

$$\pi_1(U) = \pi_1(\mathbb{D}) = \{1\} = \mathbb{Z}\rho / \rho = 1$$
$$\pi_1(\mathcal{C}) = \{1\}$$

• 

$$\pi_1(U) = \mathbb{Z} = \mathbb{Z}\rho * \mathbb{Z}\sigma / \langle \rho\sigma = 1 \rangle$$
$$\pi_1(\mathcal{C}) = \mathbb{Z}/n\mathbb{Z}$$

•  $\mathcal{F}(m,n)$

$$\pi_1(U) = \mathbb{Z}\rho_1 * \mathbb{Z}\rho_2 / \langle \rho_1\rho_2 = 1 \rangle$$
$$\pi_1 \mathcal{F}(m,n) = (\mathbb{Z}/n\mathbb{Z})\rho_1 * (\mathbb{Z}/n\mathbb{Z})\rho_2 / \langle \rho_1\rho_2 = 1 \rangle$$
$$\cong \mathbb{Z}/d\mathbb{Z} \quad \text{where } d = (m,n)$$

Prop: Let $\mathcal{C}$ connected orbifold curve, if $\mathcal{C} = \mathcal{F}(n,m)$ then the universal cover is $\mathcal{F}(m/d, n/d)$, $d=(m,n)$. Otherwise it is $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{P}^1$.

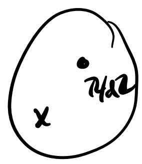
proof: We have

$$\mathcal{F}(m/d, n/d) \longrightarrow \mathcal{F}(m,n) \qquad \mathbb{Z}/d\mathbb{Z}\text{-torsor} \ \left(\text{cf. } [S^2/\mathbb{Z}/d\mathbb{Z}]\right)$$

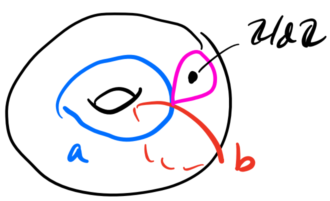
If m,n coprime, then $\pi_1 \mathcal{F}(m,n) = 1$ by previous example.

For other orbifold $\mathcal{C}$, verify that $\varphi_x : I_x \longrightarrow \pi_1(\mathcal{C}, x)$ inj $\forall x$.
If at least one hole: enough to check

$$\begin{array}{ccc} I_x & \xrightarrow{\text{id}} & \pi_1 \mathcal{C} \\ \| & & \| \\ \mathbb{Z}/d\mathbb{Z} & & \mathbb{Z}/d\mathbb{Z} \end{array}$$



If no hole, $g \geq 1$: suffices to check



$$\pi_1 \mathcal{U} = \mathbb{Z}a * \mathbb{Z}b * \mathbb{Z}c \,/\, aba^{-1}b^{-1} = c$$

$$\pi_1 \mathcal{C} = \mathbb{Z}a * \mathbb{Z}b * (\mathbb{Z}/d\mathbb{Z})c \,/\, aba^{-1}b^{-1} = c \longleftarrow \mathbb{Z}/d\mathbb{Z}$$

$$c \longleftarrow 1$$

If no hole, $g=0$, ≥ 3 orbifold pts

$$\pi_1 \mathcal{C} = (\mathbb{Z}/n_1)\rho_1 * (\mathbb{Z}/n_2)\rho_2 * (\mathbb{Z}/n_3)\rho_3 \,/\, \langle \rho_1\rho_2\rho_3 = 1 \rangle \longleftarrow \mathbb{Z}/n_i\mathbb{Z}$$

Proposition: Let $\mathcal{C}$ orbifold curve.

(i) If $|\mathcal{C}|$ not compact, then the univ cover is

- $\mathbb{C}$ if $\mathcal{C} = [\mathbb{C}/\mathbb{Z}/n]$, $\mathcal{C} = \mathbb{C} \setminus 0$, or $\mathcal{C} = \mathcal{F}(2,2) \setminus \{pt\}$
- $\mathbb{H}$ otherwise

(ii) if $|\mathcal{C}|$ is compact, then $\mathcal{C}$ hyperbolic/euclidean/spherical if

$$2g - 2 + \sum \frac{n_i - 1}{n_i} \qquad > 0 \,/\, 0 \,/ < 0$$

Secondly: uniformization for DM-stacks.

$$\mathcal{X} \longrightarrow \mathcal{C} \quad H\text{-gerbe} \quad (\text{fibers are } BH)$$

Prop: The simply connected DM-curves are $\mathbb{C}$, $\mathbb{H}$ and $\mathcal{P}(n,m)$.

pf: Let $\mathcal{X}$ simply connected. Let $\mathcal{X} \to \mathcal{C}$ H-gerbe, $\mathcal{C}$ orbifold.
Get LES:

$$\pi_2 \mathcal{C} \longrightarrow H \longrightarrow \pi_1 \mathcal{X} \longrightarrow \pi_1 \mathcal{C} \longrightarrow 1$$

$\Rightarrow$ $\mathcal{C}$ also simply connected $\Rightarrow$ $\mathcal{C} = \mathbb{C}$ or $\mathbb{H}$ or $\mathcal{P}(n,m)$ $(n,m)=1$.
If $\mathcal{C} = \mathbb{C}$ or $\mathbb{H}$ then $\mathcal{C}$ contractible $\Rightarrow \pi_2 \mathcal{C} = 1 \Rightarrow H = 1$ and $\mathcal{X} = \mathcal{C}$.
If $\mathcal{C} = \mathcal{P}(n,m)$ then one can prove that only s.c. gerbe over $\mathcal{C}$ is $\mathcal{P}(dn, dm)$.

Prop: Let $\mathcal{Y}$ orbifold, $\pi_n \mathcal{Y} = 1 \ \forall n \geq 2$, $\mathcal{X} \to \mathcal{Y}$ H-gerbe then $\tilde{\mathcal{X}} \cong \tilde{\mathcal{Y}}$ and $\mathcal{X} = [\tilde{\mathcal{Y}}/\pi_1 \mathcal{X}]$.

Cor: If $\mathcal{X} \to \mathcal{C}$ H-gerbe, $\mathcal{C}$ is euclidean/hyperbolic then so is $\mathcal{X}$ and $\mathcal{X} = [\tilde{\mathcal{C}}/\pi_1 \mathcal{X}]$.

Prop: Let $\mathcal{X} \to \mathcal{C}$ H-gerbe with $\mathcal{C}$ spherical, then so is $\mathcal{X}$.

Recall $2g - 2 + \sum \frac{n_i - 1}{n_i} < 0$

$\Rightarrow g = 0$ and ≤ 2 orbifold pts

or 3 orbifold pts w/ orders $(2,2,n)$, $(2,3,3)$, $(2,3,4)$, $(2,3,5)$

$\pi_2(\mathcal{C}) = \mathbb{Z} \Rightarrow \mathbb{Z} \cong \pi_2(\mathcal{C}) \to H \to \pi_1 \mathcal{X} \to \pi_1 \mathcal{C} \to 1$

$\Rightarrow \ker(H \to \pi_1 \mathcal{X})$ is $\mathbb{Z}/d\mathbb{Z}$ for some d.

$$\begin{tikzcd} \tilde{\mathcal{X}} \arrow[r, "H'\text{-gerbe}"] \arrow[d, "\text{cov space}"'] & \mathcal{D} \arrow[d] \\ \mathcal{X} \arrow[r, "H\text{-gerbe}"] & \mathcal{C} \end{tikzcd}$$

- $\tilde{\mathcal{X}}$ simply conn $\Rightarrow \mathcal{D}$ simply conn
- cov space $\Rightarrow \mathcal{D} \to \mathcal{C}$ cov space $\Rightarrow \mathcal{D} = \tilde{\mathcal{C}}$

$$\mathbb{Z}_d = \ker(\underset{\overset{\|}{H}}{I_x} \to \pi_1 \mathcal{X}) = \ker(I_{\tilde{x}} \to \underset{\underset{1}{\|}}{\pi_1 \tilde{\mathcal{X}}}) = H'$$

$\tilde{\mathcal{C}} = \mathcal{P}(m,n)$ $(m,n)=1$ $\Rightarrow \tilde{\mathcal{X}} = \mathcal{P}(dm, dn)$. $\square$

Rmk: $\mathcal{C} = \mathcal{F}(m,n) \Rightarrow \tilde{\mathcal{C}} = \mathcal{P}(\frac{m}{b}, \frac{n}{b})$ where $b = (m,n)$