Algebraic stacks #13

May 3, 2024

Good moduli spaces (Lecture by Jarod Alper)

Goal: Give a stack-theoretic treatment of GIT

§1: Geometric Invariant Theory (GIT)

Setup: G linearly reductive algebraic group / k field

Defn: $\mathrm{Rep}(G) \to \mathrm{Vect}_k$, $V \mapsto V^G$ is exact. ($\Leftrightarrow \mathrm{Rep}(G)$ semisimple)

finite type group scheme

Exs: - finite group if $\mathrm{char}(k) \nmid |G|$ (Maschke's thm)

- $\mathbb{G}_m$ or $\mathbb{G}_m^n$ any char.
- GL_n, SL_n, PGL_n char $= 0$
- A abelian variety (not so interesting)

Non-exs: - $\mathbb{G}_a$ in any char: $\mathbb{G}_a = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\} \curvearrowright k^2$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+by \\ y \end{pmatrix}$$

$$\begin{tikzcd}
k[x,y] \arrow[r, two heads, "y=0"] & k[x] \\
k[x,y]^{\mathbb{G}_a} \arrow[u, phantom, "\cup"] & k[x]^{\mathbb{G}_a} \arrow[u, phantom, "\cup"] \\
k[y] \arrow[u, phantom, "\shortparallel"] \arrow[r] & k[x] \arrow[u, phantom, "\shortparallel"]
\end{tikzcd}$$

not surjective $\Rightarrow$ not lin red

- B Borel $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

Main thm (affine GIT) G linearly reductive $\curvearrowright$ Spec A $/k$.

Consider π: $\underset{X}{\underset{\shortparallel}{\operatorname{Spec} A}} \longrightarrow \underset{X/\!/G}{\underset{\shortparallel}{\operatorname{Spec} A^G}}$ GIT quotient

(1) $Z \subseteq X$ closed subset $\Rightarrow$ $\pi(Z)$ closed & π surjective

(2) $Z_1, Z_2 \subseteq X$ closed G-inv $\Rightarrow$ $\pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$

If $k = \bar{k}$, $x, y \in X(k)$ then $\pi(x) = \pi(y) \Leftrightarrow \overline{Gx} \cap \overline{Gy} \neq \emptyset$

$\Rightarrow$ points of $X/\!/G$ $\Leftrightarrow$ closed orbits

(3) (universality)
$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, bend left, "G\text{-inv}"] & \\ X/\!/G \arrow[r, dashed, "\exists!"] & Y \end{tikzcd}$$
Y algebraic space

(4) (finiteness) If A f.gen$/k$ then so is A^G. (Hilbert 14^{th})

Fact: If $I \subset A^G$ then $IA \cap A^G = I$ $\Rightarrow$ A^G noetherian.

To prove finiteness of A^G:

- Choose k-alg generators $y_1, \ldots, y_k \in A$ $\rightsquigarrow$ $\langle y_1, \ldots, y_k \rangle \subseteq V \in A$ f.dim G-rep

$\Rightarrow$ $\operatorname{Sym}(V) \twoheadrightarrow A$

$\Rightarrow$ $A = \underbrace{k[x_1, \ldots, x_n]}_{G \circlearrowleft} / J$ $\leftarrow$ G-inv ideal

$k[x_1, \ldots, x_n]^G \twoheadrightarrow A^G$.

So reduces to $A = k[x_1, \ldots, x_n]$.

But $k[x_1, \ldots, x_n]^G$ graded noeth k-alg w/ constants in deg 0

$\Rightarrow$ finitely gen'd $/k$.

Example 1: $\mathbb{G}_m \curvearrowright \mathbb{A}^n$, $t.(x_1, \ldots, x_n) = (tx_1, \ldots, tx_n)$

$$
\begin{tikzcd}
\mathbb{A}^n \arrow[d] & \supset & \mathbb{A}^n \setminus 0 \arrow[d] \\
\mathbb{A}^n /\!/ \mathbb{G}_m & & \mathbb{P}^{n-1} \arrow[ll]
\end{tikzcd}
$$

$$\mathbb{A}^n /\!/ \mathbb{G}_m = \operatorname{Spec} k[x_1, \ldots, x_n]^{\mathbb{G}_m} = \operatorname{Spec} k$$

Example 2: $\mathbb{G}_m \curvearrowright \mathbb{A}^2$ $\quad t.(x,y) = (tx, t^{-1}y)$

$$
\begin{tikzcd}
\mathbb{A}^2 \arrow[d] & \supset & \mathbb{A}^2 \setminus 0 \arrow[d] \\
\mathbb{A}^2 /\!/ \mathbb{G}_m & & (\mathbb{A}^2 \setminus 0)/\mathbb{G}_m \arrow[ll]
\end{tikzcd}
$$

$$\mathbb{A}^2 /\!/ \mathbb{G}_m = \operatorname{Spec} k[x,y]^{\mathbb{G}_m} = \operatorname{Spec} k[xy] = \mathbb{A}^1 \ni a$$

$(\mathbb{A}^2 \setminus 0)/\mathbb{G}_m =$

Example 3 $\quad GL_n \curvearrowright \mathrm{Mat}_{n,n}$ (acting by conjugation)

$$\mathrm{Mat}_{n,n} \to \mathrm{Mat}_{n,n} /\!/ GL_n = \mathbb{A}^n$$

$$M \mapsto \text{char pol of } M$$

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \mapsto \text{same char pol}$$

Example 3' $\quad GL_n \curvearrowright (\mathrm{Mat}_{n,n})^k$

<u>Example 4</u>: $SL_n \curvearrowright \mathbb{A}(\mathrm{Sym}^d k^n) = \mathrm{Spec}\ \mathrm{Sym}^*(\mathrm{Sym}^d k^n)$

<u>Main thm</u> (proj GIT) G linearly reductive $\curvearrowright X \subseteq \mathbb{P}(V)$, V f.dim G-rep.

Consider $\pi : X = \mathrm{Proj}\, R \dashrightarrow \mathrm{Proj}\, R^G$ GIT quotient

$$\begin{tikzcd} X = \mathrm{Proj}\, R \arrow[r, dashed] & \mathrm{Proj}\, R^G \\ X^{ss} \arrow[u, phantom, "\cup"] \arrow[ur, "\pi"'] & \end{tikzcd}$$

<u>Defn</u>: $x \in X$ <u>semistable</u> if $\exists f \in \Gamma(X, \mathcal{O}(d))^G$, $d > 0$, s.th. $f(x) \neq 0$.

(1) $Z \subseteq X$ closed subset $\Rightarrow \pi(Z)$ closed & π surjective

(2) $Z_1, Z_2 \subseteq X$ closed G-inv $\Rightarrow \pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$

If $k = \bar{k}$, $x, y \in X(k)$ then $\pi(x) = \pi(y) \iff \overline{Gx} \cap \overline{Gy} \neq \emptyset$

$\Rightarrow$ points of $X/\!/G$ $\longleftrightarrow$ closed orbits

(3) (universality)

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "G\text{-inv}"] & \\ X/\!/G \arrow[r, dashed, "\exists !"] & Y \end{tikzcd}$$ Y algebraic space

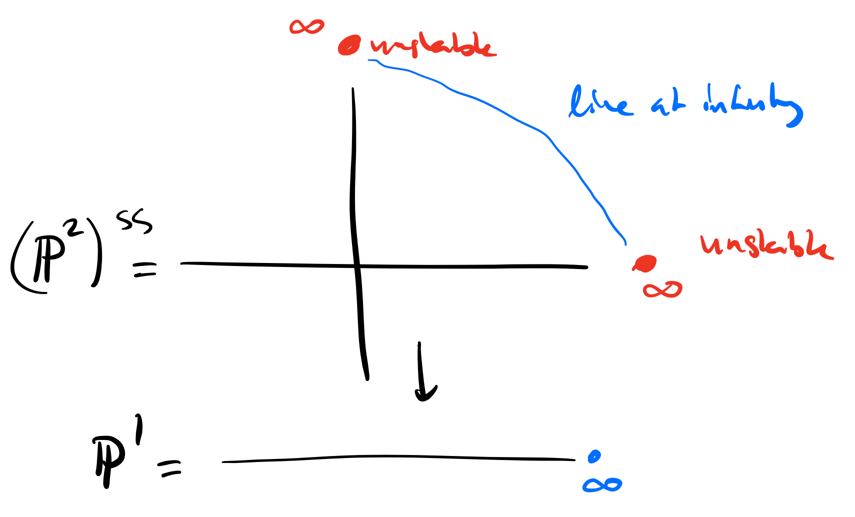
(4) (finiteness) R^G f.gen so $\mathrm{Proj}\, R^G$ projective

(Also <u>stable</u> locus $\mathbb{P}(V)^s \subset \mathbb{P}(V)^{ss}$ where action is free)

<u>Ex 2'</u>: $\mathbb{G}_m \curvearrowright \mathbb{P}^2$ $t.[x:y:z] = [tx : t^{-1}y : z]$

$k[x,y,z]^{\mathbb{G}_m} = k[xy, z]$

$\mathbb{P}^2 \setminus (\mathbb{P}^2)^{ss} = V(xy, z)$



Good: - Get projective quotient for free $\rightsquigarrow$ construct moduli

Bad: - Removes unstable locus
- Difficult to describe X^{ss}.

Hilbert–Mumford criterion (V G-rep.) $0 \neq v \in V$

$$v \in \mathbb{P}(V) \text{ not ss} \iff 0 \in \overline{Gv} \subseteq \mathbb{A}(V)$$
$$\iff \exists \lambda : \mathbb{G}_m \longrightarrow G \text{ s.th. } 0 = \lim_{t \to 0} \lambda(t).v$$

Stacks

$\mathcal{X}$ alg stack $\rightsquigarrow$ $QCoh(\mathcal{X}) = \{ \text{qcoh } \mathcal{O}_{\mathcal{X}}\text{-modules} \}$

Can think of them as $(S \to \mathcal{X}) \longmapsto F \in QCoh(S)$

Ex: $\mathcal{X} = \mathcal{M}_g$. $(S \longrightarrow \mathcal{M}_g) \longmapsto \pi_* \Omega_{\mathcal{C}/S}$

$$\Updownarrow$$

$\mathcal{C} \xrightarrow{\pi} S$

fam of curves

Fact: $f : \mathcal{X} \longrightarrow \mathcal{Y}$ qcqs gives

adjunction
$$\begin{tikzcd} QCoh(\mathcal{X}) \arrow[r, bend left, "f_*"] & QCoh(\mathcal{Y}) \arrow[l, bend left, "f^*"] \end{tikzcd}$$

Ex:

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, "p"] & {[\operatorname{Spec} A/G]} \arrow[r, "q"] \arrow[d, "\pi"] & BG \\ & \operatorname{Spec} A^G & \end{tikzcd}$$

good moduli space

* $QCoh(BG) = Rep(G)$
* $QCoh([\operatorname{Spec} A/G]) = \{A\text{-modules } M \text{ w/ } G\text{-action}\}$

$p^* M = M$ but forget G-action

$q_* M = M$ but forget A-mod str.

$\pi_* M = M^G$

Def: $\pi: \mathcal{X} \longrightarrow X$ (qcqs) is a <u>good moduli space</u> if

$\mathcal{X}$: alg stack, X: alg. space

(1) $\pi_*: QCoh(\mathcal{X}) \longrightarrow QCoh(X)$ exact

(2) $\mathcal{O}_X \longrightarrow \pi_* \mathcal{O}_{\mathcal{X}}$

Ex: G lin. red. $\curvearrowright \operatorname{Spec} A$ $\quad [\operatorname{Spec} A/G] \longrightarrow \operatorname{Spec} A^G$

<u>Good</u>: has desirable geom. prop.

<u>Bad</u>: Impossible to verify unless $\mathcal{X} = [\operatorname{Spec} A/G]$

<u>Thm/alt def</u> (A–Hall–Rydh ~'16) $\mathcal{X} \xrightarrow{f.p.} S$ qsep

$\mathcal{X}$ alg. stack, affine stab., sep. diag, any pt $x \in |\mathcal{X}|$ closed in its fiber $\mathcal{X}_s$ has linearly reductive stab.

Then $\mathcal{X} \longrightarrow X$ gms $\iff$ $\exists$

$$\begin{tikzcd} {[\operatorname{Spec} A/GL_n]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} B \arrow[r, "\text{ét}"] & X \end{tikzcd}$$

$B = A^{GL_n}$

Thm (A. '08) Let $\pi : \mathcal{X} \longrightarrow X$ gms (/ base S). Then

(1) π universally closed and surjective

(2) $Z_1, Z_2 \subseteq \mathcal{X}$ closed $\Rightarrow$ $\pi(Z_1 \cap Z_2) = \pi(Z_1) \cap \pi(Z_2)$

If $x, y \in \mathcal{X}(k)$, $k = \bar{k}$, $\pi(x) = \pi(y) \iff \overline{\{x\}} \cap \overline{\{y\}} \neq \emptyset$ in $|\mathcal{X} \times_S k|$

(3) (universality)
$$\begin{tikzcd} \mathcal{X} \arrow[d, "\pi"'] \arrow[dr, bend left] & \\ X \arrow[r, dashed, "\exists!"] & Y \end{tikzcd} \quad Y \text{ alg space}$$

(4) S noeth, $\mathcal{X}$ f.t./S $\Rightarrow$ X f.t./S and π_* preserves coh.

Lemma 1: Let $\pi : \mathcal{X} \longrightarrow X$ gms, and $\mathcal{F} \in QCoh(X)$. Then unit map $\eta : F \longrightarrow \pi_* \pi^* F$ is iso.

proof: Étale-local on X $\Rightarrow$ Can assume $X = \operatorname{Spec} A$.

Choose pres $\mathcal{O}_X^J \longrightarrow \mathcal{O}_X^I \longrightarrow F \longrightarrow 0$

$\Rightarrow$ $\mathcal{O}_{\mathcal{X}}^J \longrightarrow \mathcal{O}_{\mathcal{X}}^I \longrightarrow \pi^* F \longrightarrow 0$ (π^* right-exact)

$$\begin{tikzcd} \pi_* \mathcal{O}_{\mathcal{X}}^J \arrow[r] \arrow[d, equal, "(2)"] & \pi_* \mathcal{O}_{\mathcal{X}}^I \arrow[r] \arrow[d, equal, "(2)"] & \pi_* \pi^* F \arrow[r] & 0 \\ \mathcal{O}_X^J & \mathcal{O}_X^I & & \end{tikzcd}$$
$\Rightarrow$ (π_* exact) (1)

$\Rightarrow$ $F = \pi_* \pi^* F$ $\square$

Lemma 2: Consider cartesian

$$\begin{tikzcd} \mathcal{X}' \arrow[r, "g'"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] & \text{(stacks)} \\ X' \arrow[r, "g"'] & X & \text{(alg spaces)} \end{tikzcd}$$

(1) π gms $\Rightarrow$ π' gms

(2) π' gms, g fppf $\Rightarrow$ π gms.

proof: (2) flat base change $\Rightarrow$ $\underbrace{g^*}_{\substack{\text{faithfully}\\ \text{exact}}}\pi_* \longrightarrow \underbrace{\pi'_* g'^*}_{\text{exact by ass}}$ iso $\Rightarrow$ π_* exact

flat base change $\Rightarrow$ $\underset{\substack{\| \\ \mathcal{O}_{X'}}}{g^*\pi_*\mathcal{O}_{\mathcal{X}}} \longrightarrow \underset{\substack{\| \\ \pi'_*\mathcal{O}_{\mathcal{X}'}}}{\pi'_* g'^*}$ iso $\Rightarrow$ $\mathcal{O}_{X'} \xrightarrow{\cong} \pi'_*\mathcal{O}_{\mathcal{X}'}$

(1) B/c of (2) can assume $X' = \operatorname{Spec} A' \xrightarrow{g} X = \operatorname{Spec} A$

$$\underbrace{g_*}_{\substack{\text{faithfully}\\ \text{exact}\\ \text{(b/c } g \text{ affine)}}}\pi'_* = \underbrace{\pi_* g'_*}_{\substack{\text{exact}\\ (\text{ass} + g' \text{ aff})}} \quad \Rightarrow \quad \pi'_* \text{ exact}$$

Study $\mathcal{O}_{X'} \xrightarrow{\eta} \pi'_*\mathcal{O}_{\mathcal{X}'}$. We have

$$\begin{tikzcd} g_*\mathcal{O}_{X'} \arrow[r, "g_*\eta"] \arrow[rrrr, bend right, "="'] & g_*\pi'_*\mathcal{O}_{\mathcal{X}'} \cong \pi_* g'_* \pi'^*\mathcal{O}_{X'} \overset{g \text{ affine}}{\cong} \pi_*\pi^* g_*\mathcal{O}_{X'} & & & \overset{\text{Lemma 1}}{\cong} g_*\mathcal{O}_{X'} \end{tikzcd}$$

$\Rightarrow$ $g_*\eta$ iso $\Rightarrow$ η iso b/c g_* conservative. $\square$

pf of Thm: (1) For surjective: let $x \in |X|$, then

$$\begin{tikzcd} \mathcal{X}_x \arrow[r] \arrow[d, "\pi_x"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "\pi"] \\ \operatorname{Spec} k(x) \arrow[r] & X \end{tikzcd}$$

π_x also gms $\Rightarrow$ $k(x) = \Gamma(\mathcal{X}_x) \neq 0$ $\Rightarrow$ $\mathcal{X}_x \neq \emptyset$

For univ closed, enough to prove close. Let $\mathcal{Z} \subseteq \mathcal{X}$ closed substack defined by $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}} \twoheadrightarrow \mathcal{O}_{\mathcal{Z}}$.

$$\begin{tikzcd} \mathcal{Z} \arrow[r, hook] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circ"] & \mathcal{X} \arrow[d, "\pi"] \\ \overline{\pi(\mathcal{Z})} \arrow[r, hook] & X \end{tikzcd}$$

schematic image $\rightarrow$

$\overline{\pi(\mathcal{Z})}$ defined by $\pi_* \mathcal{I}$

$$\left(0 \to \pi_* \mathcal{I} \to \underset{\mathcal{O}_X}{\underset{\shortparallel}{\pi_* \mathcal{O}_{\mathcal{X}}}} \to \pi_* \mathcal{O}_{\mathcal{Z}} \to 0\right)$$

Claim: π' is a gms

$\Rightarrow \pi'$ surjective $\Rightarrow$ $\pi(\mathcal{Z}) = \overline{\pi(\mathcal{Z})}$ closed.

(2) Need to show that $\pi_*(\mathcal{I}_1) + \pi_*(\mathcal{I}_2) \hookrightarrow \pi_*(\mathcal{I}_1 + \mathcal{I}_2)$ surjective.
for any $\mathcal{I}_1, \mathcal{I}_2 \subset \mathcal{O}_{\mathcal{X}}$.

$$0 \to \mathcal{I}_1 \to \mathcal{I}_1 + \mathcal{I}_2 \to \mathcal{I}_2 / \mathcal{I}_1 \cap \mathcal{I}_2 \to 0$$

$$\begin{tikzcd}[column sep=small] & & & \pi_*(\mathcal{I}_2) \arrow[d] \arrow[dr, two heads] & \\ \Rightarrow \quad 0 \arrow[r] & \pi_* \mathcal{I}_1 \arrow[r] & \pi_*(\mathcal{I}_1 + \mathcal{I}_2) \arrow[r] & \pi_*(\mathcal{I}_2)/\pi_*(\mathcal{I}_1 \cap \mathcal{I}_2) \arrow[r] & 0 \end{tikzcd}$$

β

$\beta - \alpha \in \pi_* \mathcal{I}_1$

$\alpha \mapsto \bar{\alpha}$

$\beta - \alpha \mapsto 0$

$\Rightarrow \alpha = \beta + (\beta - \alpha)$

(3)
$$\begin{tikzcd} \mathfrak{X} \arrow[d, "\pi"'] \arrow[dr, "\phi"] & \\ X \arrow[r, dashed] & Y \end{tikzcd}$$

Case Y affine:

$$\begin{aligned} \mathrm{Mor}(\mathfrak{X}, Y) &= \mathrm{Hom}(\Gamma(Y), \Gamma(\mathfrak{X})) \\ &= \mathrm{Hom}(\Gamma(Y), \Gamma(X)) \\ &= \mathrm{Mor}(X, Y) \end{aligned}$$

Case Y scheme:

$Y = \bigcup_i Y_i$ affine cover $\Rightarrow$ $\mathfrak{X} = \bigcup_i \mathfrak{X}_i$ $\quad \mathfrak{X}_i = \phi^{-1}(Y_i)$

$\mathfrak{X} \setminus \mathfrak{X}_i$ closed $\Rightarrow$ $\pi(\mathfrak{X} \setminus \mathfrak{X}_i)$ closed

$\mathfrak{X} \setminus \mathfrak{X}_i \subseteq \pi^{-1}(\pi(\mathfrak{X} \setminus \mathfrak{X}_i))$ $\Rightarrow$
$$\begin{tikzcd} \mathfrak{X} \setminus \pi^{-1}(\pi(\mathfrak{X} \setminus \mathfrak{X}_i)) \subset \mathfrak{X}_i \arrow[r] \arrow[d, "\text{gms}"] & Y_i \\ X \setminus \pi(\mathfrak{X} \setminus \mathfrak{X}_i) \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$
(affine case)

$\bigcap_i (\mathfrak{X} \setminus \mathfrak{X}_i) = \emptyset$ $\Rightarrow$ $\bigcap_i \pi(\mathfrak{X} \setminus \mathfrak{X}_i) = \emptyset$ $\Rightarrow$ $X = \bigcup X \setminus \pi(\mathfrak{X} \setminus \mathfrak{X}_i)$
open cov.

(4) X noeth: $\mathcal{J} \subseteq \mathcal{O}_X$ $\Rightarrow$
$$\begin{tikzcd} \pi^{-1}(V(\mathcal{J})) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \mathfrak{X} \arrow[d] \\ V(\mathcal{J}) \arrow[r, hook] & X \end{tikzcd}$$

$\Rightarrow$ $\mathcal{J} = \pi_*(\pi^{-1}\mathcal{J} \cdot \mathcal{O}_{\mathfrak{X}})$

Gives X noeth as before.

For finite type: algebra fact:

$$\begin{tikzcd} U \arrow[d, "\text{qc}"', "\text{univ. subm}"'] \arrow[dr, "\text{finite type}"] & \\ X \arrow[r] & S \\ \text{noeth} & \text{noeth} \end{tikzcd} \implies X \to S \text{ finite type}$$

Apply this to:

$$\begin{tikzcd} U \arrow[d, "\text{sm. pres.}"'] \arrow[ddr] & \\ \mathcal{X} \arrow[d, "\pi", "\text{univ cl + surj}"'] & \\ X \arrow[r] & S \end{tikzcd}$$

<u>Missing piece in char p</u>: local structure thm

$$\begin{tikzcd} {[\operatorname{Spec} A/GL_n]} \arrow[dr] & \\ & \mathcal{X} \end{tikzcd}$$

$$w \mapsto x$$