Algebraic stacks #12

Apr 24, 2024

§ Toric varieties (Lecture by Darius)

Def: • The n-torus is $\mathbb{T}^n := \mathbb{G}_m^n = (\mathbb{C}^\times)^n$.

• A toric variety is an irreducible algebraic variety X w/ a torus $\mathbb{T}^n \cong T \subseteq X$ as a Zariski-open subset and such that $T \curvearrowright T$ extends to $T \curvearrowright X$.

To $\mathbb{T}^n$ associate two lattices:

$M = \mathrm{Hom}(\mathbb{T}^n, \mathbb{C}^\times)$ character lattice

$N = \mathrm{Hom}(\mathbb{C}^\times, \mathbb{T}^n)$ cocharacter lattice (1-parameter subgroups)

Rmk: $M \cong \mathbb{Z}^n \cong N$ but really M and N are dual.

Construction:

1) Group algebra $\mathbb{C}[M] = \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $\mathbb{T} \overset{n}{=} \operatorname{Spec} \mathbb{C}[M]$

Idea: submonoid $M' \subset M$ gives $\mathbb{C}[M'] \subset \mathbb{C}[M] \Rightarrow \mathbb{T}^n \subset \operatorname{Spec} \mathbb{C}[M']$.

2) Take a (rational, convex, polyhedral) cone $\sigma \subseteq N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$

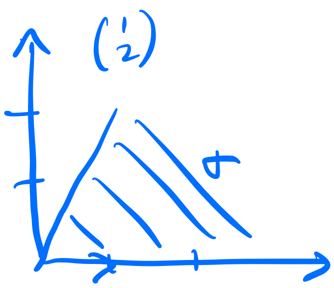
i.e., $\sigma = \mathbb{R}_{\geq 0} \cdot A$, $A \subseteq N$ finite

$\Rightarrow S_\sigma := \sigma^\vee \cap M$ f.g. submon of M

$\Rightarrow \mathbb{C}[S_\sigma] \subset \mathbb{C}[M]$

$\Rightarrow \mathbb{T}^n \subset X_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma]$

Ex: $A = \left\{ \binom{1}{0}, \binom{1}{2} \right\}$



and $\mathbb{C}[S_\sigma] \longrightarrow \mathbb{C}[S_\sigma] \otimes \mathbb{C}[M]$ gives torus action.

$\chi \longmapsto \chi \otimes \chi$

3) Next, take a <u>lattice fan</u>, i.e., a collection Σ of cones σ $\sigma \subseteq N_{\mathbb{R}}$ s.th.

a) each $\sigma \in \Sigma$ strongly convex, rational, polyhedral

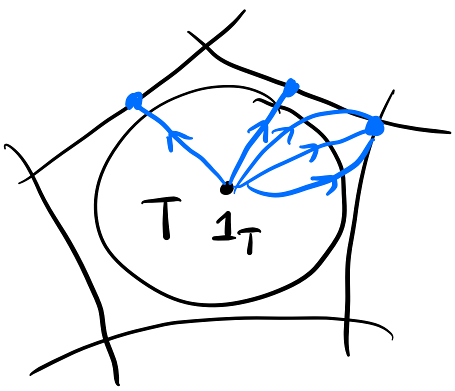
b) for each $\sigma \in \Sigma$, each face of σ is also in Σ

c) for $\sigma_1, \sigma_2 \in \Sigma$, $\sigma_1 \cap \sigma_2$ is a face of both σ_1, σ_2.

Then each $\sigma \in \Sigma$ defines an affine $X_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$ and these glue via $X_{\sigma_1 \cap \sigma_2} \begin{smallmatrix} \subset X_{\sigma_1} \\ \subset X_{\sigma_2} \end{smallmatrix}$ to a toric variety X_Σ.

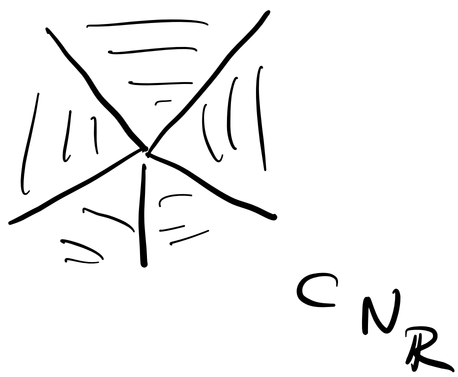
<u>Reverse direction:</u> Let X toric variety. Consider $\lambda : \mathbb{C}^\times \longrightarrow X$

Does $\lim_{t \to 0} \lambda(t) \cdot 1_T \in X$?



X

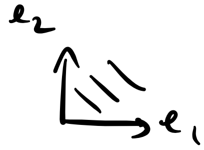
$$\lambda \mapsto \frac{d\lambda}{dt}(1) \quad \longrightarrow$$

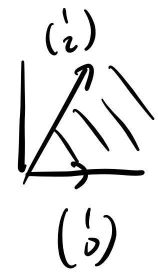
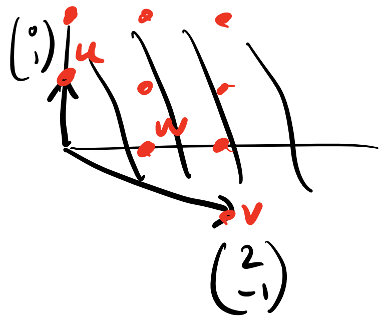


convergency fan Σ_X

<u>Thm</u> There is an equivalence of categories

$$\{\text{normal toric varieties}\} \longleftrightarrow \{\text{lattice fans}\}$$

Ex: 1) $\sigma^\vee =$  $\mathbb{C}[S_\sigma] = \mathbb{C}[x,y]$

2) $\sigma =$  $\rightsquigarrow$ $\sigma^\vee =$ 

$$\mathbb{C}[S_\sigma] = \mathbb{C}[u,v,w]/(uv-w^2) \cong \mathbb{C}[x^2,y^2,xy]$$
$$= \mathbb{C}[x,y]^{\mu_2}$$

(μ_2 acts via $x \mapsto -x$, $y \mapsto -y$)

Rmk: $\mathbb{C}[S_\sigma]$ singular b/c (rays) $\cap N$ don't generate N.

§ Toric stacks

Def (Geraschenko–Satriano)

1) A *toric stack* is an Artin stack of the form $[X/G]$ where X toric variety and $G \subseteq T \subseteq X$ w/ action of T/G.

2) A *non-strict toric stack* is an Artin stack $[Z/G]$ where $Z \subseteq X$ is an integral T-invariant subvariety.

Corresponding notion of fans for toric stacks?

Def: 1) A <u>stacky fan</u> is a pair (Σ, β) where Σ lattice fan in a lattice L and $\beta : L \longrightarrow N$ a lattice morphism with $|\operatorname{coker} \beta| < \infty$.

2) A <u>non-strict stacky fan</u> is a pair (Σ, β) as above but N any f.g. abelian group

Construction: Let (Σ, β) be a <u>stacky fan</u>. Let $X = X_\Sigma$.
coker β finite $\Rightarrow$ $\beta^\vee : N^\vee \longrightarrow L^\vee$ injective. Induces a morphism of tori:

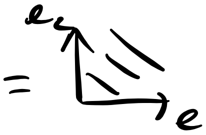
$$T_\beta : T_L \longrightarrow T_N \qquad \left(T_L = \operatorname{Spec} \mathbb{C}[L^\vee] \right)$$

which is surjective. Then $T_L \subset X_\Sigma$ and $G_\beta := \ker(T_\beta) \subseteq T_L$.
We let the toric stack associated to (Σ, β) be:

$$\mathcal{X}_{(\Sigma,\beta)} := [X_\Sigma / G_\beta]$$

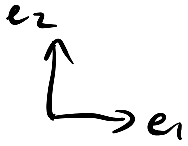
Examples:

1) If $\beta = \mathrm{id}$, then $G_\beta = \{1\}$, $\mathcal{X}_{(\Sigma, \mathrm{id})} = X_\Sigma$ toric variety

2) $\Sigma =$  , $\beta : \underset{L}{\mathbb{Z}^2} \xrightarrow{\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}} \underset{N}{\mathbb{Z}^2}$, $\beta^\vee : N^\vee \xrightarrow{\begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}} L^\vee$

$$T_\beta : \mathbb{G}_m^2 \longrightarrow \mathbb{G}_m^2, \quad (s,t) \longmapsto (st, t^2)$$
$G_\beta = \ker(T_\beta) = \mu_2$. Then $\mathcal{X}_{(\Sigma,\beta)} = [\mathbb{A}^2/\mu_2]$.

Note that $\mathcal{X}_{(\Sigma,\beta)}$ smooth. Gives a crepant res of $X_{\Sigma'}$, $\Sigma' =$ 

3) $\Sigma =$ 

$X_\Sigma = \mathbb{A}^2 \setminus \{0,0\}$

$\beta : \mathbb{Z}^2 \xrightarrow{(1\ -1)} \mathbb{Z}$ $\qquad$ $T_\beta : \mathbb{G}_m^2 \longrightarrow \mathbb{G}_m$

$\qquad\qquad\qquad\qquad\qquad\qquad (s,t) \longmapsto st^{-1}$

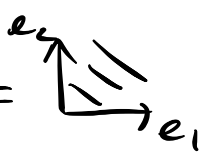
$\ker T_\beta = \{(t,t)\} \subseteq \mathbb{G}_m^2$
$\cong \mathbb{G}_m$

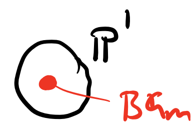
$\mathcal{X}_{(\Sigma,\beta)} = [\mathbb{A}^2 \setminus \{(0,0)\} / \mathbb{G}_m] = \mathbb{P}^1$

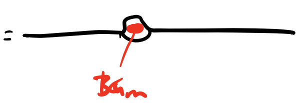
4) Σ as above, $\beta : \mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$, $\ker T_\beta = \{(t, t^{-1})\} \subseteq \mathbb{G}_m^2$

$\mathcal{X}_{(\Sigma,\beta)} = [\mathbb{A}^2 \setminus \{(0,0)\} / \underset{1,-1}{\mathbb{G}_m}] = \mathbb{A}^1 \underset{\mathbb{A}^1 \setminus 0}{\cup} \mathbb{A}^1 =$ 

non-sep $\mathbb{A}^1$.

5) $\Sigma =$ 

, $\beta : \mathbb{Z}^2 \xrightarrow{(1\ -1)} \mathbb{Z}$, $\mathcal{X}_{(\Sigma,\beta)} = [\mathbb{A}^2 / \underset{1,1}{\mathbb{G}_m}] =$ 

6) Σ as above, $\beta : \mathbb{Z}^2 \xrightarrow{(1\ 1)} \mathbb{Z}$, $\mathcal{X}_{(\Sigma,\beta)} = [\mathbb{A}^2 / \underset{1,-1}{\mathbb{G}_m}] =$ 

Second hour: (lecture by Ludwig)

Thm: A pre-variety X containing a torus $T \subseteq X$ such that the action $T \curvearrowright T$ extends to $T \curvearrowright X$ is of the form X_Σ if and only if

(1) X is normal

(2) X is separated

Thm: An Artin stack $\mathfrak{X}$ (over an alg. closed field of char 0) with an open dense substack $T \subseteq \mathfrak{X}$ for which $T \curvearrowright T$ extends to $T \curvearrowright \mathfrak{X}$ is a toric stack if and only if

(1) $\mathfrak{X}$ is normal

(2) The diagonal $\mathfrak{X} \xrightarrow{\Delta} \mathfrak{X} \times \mathfrak{X}$ is affine

(3) Every stabilizer of $\mathfrak{X}$ is linearly reductive

[(4) $\mathfrak{X}$ is of "global type"] (automatic)

Examples & non-examples

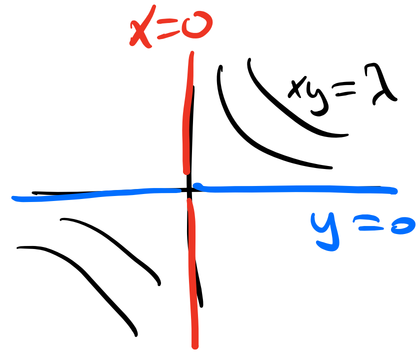
- (Δ not affine) $X = \mathbb{A}^2 \cup_{\mathbb{A}^2 \setminus 0} \mathbb{A}^2$

$$\begin{array}{ccc} \Delta : X & \longrightarrow & X \times X \\ \cup\mid & & \cup\mid \\ \Delta^{-1}(\mathbb{A}^4) & & \mathbb{A}^4 \text{ where } 0 \\ \shortparallel & & \text{is } (0_1, 0_2) \\ \mathbb{A}^2 \setminus 0 & & \\ \text{not affine} & & \end{array}$$

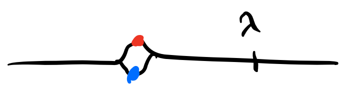
- $X = \mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1$ non-separated but is a toric stack! (= Ex 4 above)

$$G = \{(t,t^{-1})\} \subset \underset{\overset{\parallel}{T}}{\mathbb{G}_m^2} \subset \mathbb{A}^2 \setminus \{0\}$$

Orbits of $G \cong \mathbb{G}_m$ on $\mathbb{A}^2\setminus\{0\}$



$$[\mathbb{A}^2\setminus\{0\}/G] =$$



$$= \mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1$$

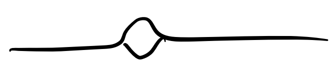
- $X = [M_2/GL_2]$ action by left multiplication

$$\mathrm{stab}\left(\begin{bmatrix}1&0\\0&0\end{bmatrix}\right) = \left\{\begin{bmatrix}a&b\\c&d\end{bmatrix} \in GL_2 : \begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix} = \begin{bmatrix}1&0\\0&0\end{bmatrix}\right\}$$

$$= \left\{\begin{bmatrix}1&b\\0&d\end{bmatrix} : d \neq 0\right\} \cong \mathbb{G}_m \ltimes \mathbb{G}_a$$ not reductive

- (action does not extend)

$$X = \mathbb{A}^1 \cup_{\mathbb{A}^1\setminus 0} \mathbb{A}^1 =$$



$\mathbb{Z}/2 \curvearrowright X$ $\quad t \longmapsto -t$ & swaps two origins.

normal non-sep algebraic space →

$$\begin{array}{ccc}
\mathcal{X} = [X/\mathbb{Z}/2] & \xleftarrow{\text{étale}} & X \\
\cup & & \cup \\
\mathbb{G}_m \cong [\mathbb{G}_m/\mathbb{Z}/2\mathbb{Z}] & \longleftarrow & \mathbb{G}_m \\
t^2 & \longleftarrow & t
\end{array}$$

If $\mathbb{G}_m \curvearrowright \mathbb{G}_m$ extends to $\mathbb{G}_m \curvearrowright \mathcal{X}$ we get:

$$\begin{tikzcd}
T_0\mathcal{X} \arrow[d, equal] & T_0 X \arrow[l, "\cong"'] \arrow[d, equal] \\
\mathbb{G}_m \curvearrowright k & \mathbb{G}_m \curvearrowright k
\end{tikzcd}$$

$$t.v = tv \qquad\qquad t.v = tv$$

$$\begin{tikzcd}
\mathbb{G}_m & \mathbb{G}_m \arrow[l] \\
t^{\pm 2} & t \arrow[l, mapsto]
\end{tikzcd}$$

... $\rightsquigarrow$ contradiction.