Algebraic stacks #11

Apr 17, 2024

Root stacks: (Torger)

<u>Ref</u> (Cadman '07 "Using stacks to impose tangency conditions on curves")

§ Root of line bundles (root gerbes)

<u>Motivation</u>: $L \in \operatorname{Pic} X$. Does there exist $M \in \operatorname{Pic} X$ s.th. $M^{\otimes n} \cong L$?

Not always but root stacks provide this:

$$X' \xrightarrow{p} X, \quad L' \in \operatorname{Pic} X' \quad \text{s.th. } p^* L \cong (L')^{\otimes n}$$

<u>Recall</u>:

$$\{X \longrightarrow B\mathbb{G}_m\} = \left\{ \begin{array}{c} E \\ \downarrow \\ X \end{array} \ \text{principal } \mathbb{G}_m\text{-bundle} \right\} = \{ L \in \operatorname{Pic} X \}$$

<u>Def</u> (Root of $L \in \operatorname{Pic} X$) We have $\mathbb{G}_m \xrightarrow{\wedge r} \mathbb{G}_m \rightsquigarrow B\mathbb{G}_m \xrightarrow{\overline{\theta}_r} B\mathbb{G}_m$

$$\begin{array}{ccc} L & \longmapsto & L^{\otimes r} \\ \in & & \in \\ B\mathbb{G}_m(X) & & B\mathbb{G}_m(X) \end{array}$$

This gives:

$$\begin{tikzcd} X_{L,r} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B\mathbb{G}_m \arrow[d, "\overline{\theta}_r"] \\ X \arrow[r, "{[L]}"'] & B\mathbb{G}_m \end{tikzcd}$$

Different notations

$$X_{L,r} = r^{th} \text{ root stack of } L \ (\text{"root gerbe"}) = \underbrace{\sqrt[r]{L} = X(\sqrt[r]{L})}$$

$$X_{L,r}(T) = \{ T \xrightarrow{p} X, \ L' \in \operatorname{Pic} T : \ p^* L \cong (L')^{\otimes n} \}$$

<u>Ex</u>:

$$\begin{tikzcd} \sqrt[n]{\mathcal{O}_{\mathbb{A}^1}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B\mathbb{G}_m \arrow[d, "\overline{\theta}_n"] \\ \mathbb{A}^1 \arrow[r, "{[\mathcal{O}]}"] & B\mathbb{G}_m \end{tikzcd}$$

Then $\sqrt[n]{\mathcal{O}_{\mathbb{A}^1}} = \mathbb{A}^1 \times B\mu_n$

§ Generalized Cartier divisors

Def A generalized Cartier div is (L, s) where $L \in \mathrm{Pic}\, X$, $s \in \Gamma(X, L)$.

Ex: D eff. Cartier div $\rightsquigarrow (\mathcal{O}(D), s_D)$ gen. Cartier div.

Prop: $[\mathbb{A}^1/\mathbb{G}_m](T) = \{(L, s)\}$

Olsson: Constructs a fibered category with

obj: $(T, (L, s))$

mor: $(T', (L', s')) \longrightarrow (T, (L, s))$ is $\begin{cases} g : T' \to T \\ g^b : (L', s') \xrightarrow{\cong} (g^*L, g^*s) \end{cases}$

More intuitively: $[\mathbb{A}^1/\mathbb{G}_m](T) = \left\{ \begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\mathbb{G}_m\text{-bundle}"'] & \mathbb{A}^1 \\ T & \end{tikzcd} \right\}$ $\qquad k[t] \longrightarrow \bigoplus_{n \in \mathbb{Z}} L^n$

$\rightsquigarrow P \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1$ $\rightsquigarrow L := P \overset{\mathbb{G}_m}{\times} \mathbb{A}^1 \xrightarrow{s} \mathbb{A}^1$

$x, \lambda \longmapsto \lambda x$ $\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad P \times \mathbb{A}^1/\mathbb{G}_m$ $\qquad \mu.(x, \lambda) = (x\mu^{-1}, \mu\lambda)$

$\begin{matrix} L \\ \downarrow \\ T \end{matrix}$ is the associated line bundle and $s : L \to \mathbb{A}^1 \iff \mathcal{L}^\vee \to \mathcal{O}_T \iff \mathcal{O}_T \to \mathcal{L}$ $\qquad (L = \mathrm{Spec}\, \mathrm{Sym}(\mathcal{L}^\vee))$

$\theta_r : [\mathbb{A}^1/\mathbb{G}_m] \longrightarrow [\mathbb{A}^1/\mathbb{G}_m]$ given by $\quad \mathbb{A}^1 \xrightarrow{\wedge r} \mathbb{A}^1$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mathbb{G}_m \xrightarrow{\wedge r} \mathbb{G}_m$

$(L, s) \in [\mathbb{A}^1/\mathbb{G}_m](T)$ $\qquad\qquad\qquad\qquad a \longmapsto a^r$

$\Rightarrow \theta_r(L, s) = (L^{\otimes r}, s^{\otimes r})$

§ Root stacks of (generalized) Cartier divisors

<u>Def</u>: Root stack $X_{(L,s,r)}$ of generalized Cartier divisor (L,s):

$$\begin{tikzcd} X_{(L,s,r)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\theta_r"] \\ X \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]} \end{tikzcd}$$

<u>Different notations</u>: $X_{(L,s,r)} = \sqrt[r]{(L,s)} = X(\sqrt[r]{L,s})$

$$X_{(L,s,r)}(T) = \left\{ \begin{array}{l} g : T \to X,\ (M,\lambda) \text{ gen. cd on } T,\ \delta : M^{\otimes r} \xrightarrow{\cong} L \\ \qquad\qquad\qquad\qquad\qquad \text{s.th. } \delta \circ \lambda^r = g^* s \end{array} \right\}$$

<u>Ex</u>: $\underset{\text{Spec } k[x]}{\underset{\shortparallel}{X = \mathbb{A}^1}}$, $L = \mathcal{O}_X$, $s = x$. Then $X_{\mathcal{O}_X, x, r} = [\mathbb{A}^1/\mu_r]$

<u>Lemma</u> (Olsson Exc. 10F) $G \xrightarrow{\varphi} H$, $G \curvearrowright X$, $H \curvearrowright Y$
$X \xrightarrow{f} Y$, φ-equivariant, then if φ surjective

$$\begin{tikzcd} {[X/K]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[X/G]} \arrow[d] \\ Y \arrow[r] & {[Y/H]} \end{tikzcd}$$

where $K = \ker(\varphi)$.

Apply Lemma with $\varphi = \theta_r : \mathbb{G}_m \to \mathbb{G}_m$, $\ker(\theta_r) = \mu_r$ to deduce example.

Prop (Olsson, 10.3.10 (ii)) Let $(\mathcal{O}_X, s)$ gcd on X. Then

$$X_{(\mathcal{O}_X, s, r)} = \left[\operatorname{Spec}_X \mathcal{O}_X[t]/(t^r - s) \,/\, \mu_r \right]$$

pf:

$$\begin{tikzcd}
X \times_{\mathbb{A}^1} [\mathbb{A}^1/\mu_r] \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mu_r]} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "\theta_r"] \\
X \arrow[r, "s"] & \mathbb{A}^1 \arrow[r, "{(\mathcal{O}, x)}"] & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}$$

$\square$

Cor $X_{(L,s,r)} \to X$ iso on $U = \{ s \neq 0 \}$.

§ Comparison

$X_{(L,r)}$ root gerbe vs $X_{(L,s,r)}$ root stack of gr. div.

$$\begin{tikzcd}
X_{(L,r)} \arrow[r] \arrow[d, hook] & B\mathbb{G}_m \arrow[d, "\text{inf. thickening}"] \arrow[dr, bend left, "0"] & \\
\text{root stack} \quad X_{(L,0,r)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[r] \arrow[d] \arrow[ddr, phantom, "\square"] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[dd, "\theta_r"] \\
\text{root gerbe} \quad X_{(L,r)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B\mathbb{G}_m \arrow[d, "\bar{\theta}_r"] & \\
X \arrow[r, "{(L,0)}"] & B\mathbb{G}_m \arrow[r, "0"] & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}$$

§ Root stacks of usual effective divisors

__Def:__ $X(\sqrt[r]{D}) := X_{(\mathcal{O}(D), s_D, r)}$

$$
\begin{tikzcd}
E = D_{(\mathcal{O}(D)|_D, r)} \arrow[rr] \arrow[dr] \arrow[dd, "\text{root gerbe}"'] & & B\mathbb{G}_m \arrow[dr, "0"] \arrow[dd, "\bar{\theta}_r"] & \\
& X(\sqrt[r]{D}) \arrow[rr, "(E, s_E)"] \arrow[dd, "\pi"] & & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[dd, "\theta_r"] \\
D \arrow[rr] \arrow[dr, bend right] & & B\mathbb{G}_m \arrow[dr, "0"] & \\
& X \arrow[rr, "(D, s_D)"'] & & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}
$$

__NB!__ Commutative squares left/right not cartesian.

E "exceptional divisor"

$rE = \pi^{-1}(D)$

__Facts:__

- X, D smooth $\Rightarrow$ $X_{D,r}$ smooth
- $D = D_1 + D_2 + \dots + D_n$ snc divisor

$$X \xrightarrow{((\mathcal{O}(D_i), s_{D_i}))_{i=1}^{n}} [\mathbb{A}^1/\mathbb{G}_m]^n = [\mathbb{A}^n/\mathbb{G}_m^n]$$

$$\underset{\text{smooth}}{X(\sqrt[r]{D_1}, \sqrt[r]{D_2}, \dots, \sqrt[r]{D_n})} \longrightarrow X \quad \left(\begin{array}{c} \text{iterated root or fiber prod} \\ X(\sqrt[r]{D_1}) \times_X \dots \times_X X(\sqrt[r]{D_n}) \end{array} \right)$$

- Δ_X finite $\Rightarrow$ $\Delta_{X_{(L,s,r)}}$ finite

- X DM, $r \in \mathcal{O}_X^\times$ $\Rightarrow$ $X_{(L,s,r)}$ DM

(2nd hour: Jon-Magnus)

§1 Root stacks as global quotients

(ref: Quek–Rydh '22 "Weighted blow-ups")

§ Stack-theoretic Proj

$R = \bigoplus_{n \geq 0} R_n$ quasi-coh. $\mathbb{Z}$-graded $\mathcal{O}_X$-mod $\qquad R_+ = \bigoplus_{n \geq 1} R_n$

$\mathbb{Z}$-grading $\Rightarrow$ $\mathbb{G}_m$-action on $\operatorname{Spec} R$.

$$\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{\alpha} \operatorname{Spec} R$$

$$\begin{array}{ccc} R[t^{\pm 1}] = \mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}} R & \longleftarrow & R \\ r t^n & \longleftarrow & r \in R_n \end{array}$$

Given a point $x: \operatorname{Spec} k \to \operatorname{Spec}_X R$, get stabilizer:

$$\begin{tikzcd} G_x \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \operatorname{Spec} k \arrow[d, "(x,x)"] \\ \mathbb{G}_m \times \operatorname{Spec}_X R \arrow[r, "(\alpha, pr_2)"'] & \operatorname{Spec}_X R \times \operatorname{Spec}_X R \end{tikzcd}$$

Note $G_x \subset \mathbb{G}_m$ so either $G_x = \mathbb{G}_m$ or $G_x = \mu_d$ for some d.

Lemma (QR 1.1.2) $G_x^\vee = \mathbb{Z}/\langle d : x \notin V(R_d) \rangle$

In particular, $G_x = \mathbb{G}_m \iff x \in V(R_+)$.

$$\mu_r \subseteq G_x \iff x \in V(R_n) \ \forall n : r \nmid n.$$

pf: Assume X is affine. Then $\mathrm{Spec}_X R = \mathrm{Spec}\, R$.

and $x : \mathrm{Spec}\, k \to \mathrm{Spec}_X R$ given by $\varphi_x : R \to k$.

Then x μ_r-equi (trivial action on k) $\iff$ φ_x $\mathbb{Z}/r\mathbb{Z}$-graded.

$\iff \varphi_x(R_d) = 0 \ \forall d : r \nmid d.$ $\square$

Def: The stack-theoretic proj of R is

$$\mathcal{P}roj_X(R) = [\mathrm{Spec}_X R \setminus V(R_+) / \mathbb{G}_m]$$

Rmk: If R gen'd in deg 1, then $V(R_+) = V(R_1)$ then (by lemma) $\mathbb{G}_m \curvearrowright \mathrm{Spec}_X R \setminus V(R_+)$ is free so

$$\mathrm{Proj}_X(R) = \mathcal{P}roj_X(R)$$

$$\begin{tikzcd}
\mathrm{Spec}_X R \setminus V(R_+) \arrow[r, "\mathbb{G}_m\text{-equi}"] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & * \arrow[d, "\text{univ } \mathbb{G}_m\text{-torsor}"] \\
\mathcal{P}roj_X R \arrow[r] & X \times B\mathbb{G}_m \arrow[r] & B\mathbb{G}_m
\end{tikzcd}$$

Ex: X scheme, $\mathcal{L}$ line bundle on X.

$$\mathbb{V}(\mathcal{L}) = \operatorname{Spec}_X(\operatorname{Sym}\mathcal{L}^\vee) = \operatorname{Spec}_X(\underbrace{\mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \mathcal{L}^{\vee 2} \oplus \cdots}_{R})$$

$V(R_+)$ = zero-section of line bundle

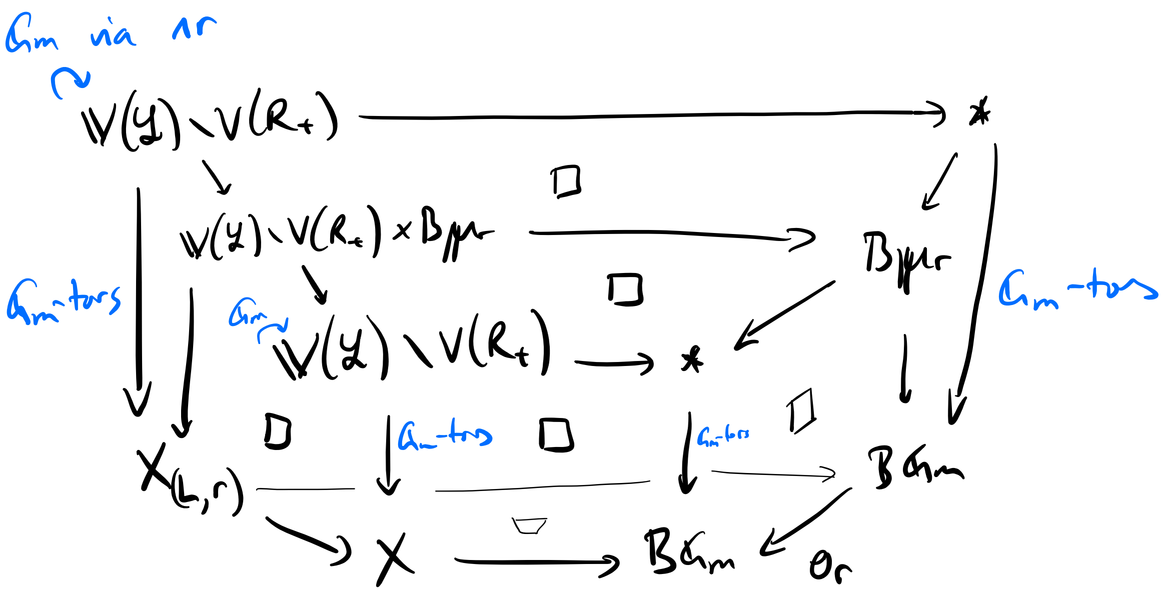
$\mathbb{V}(\mathcal{L}) \setminus V(R_+)$ is corresponding $\mathbb{G}_m$-torsor

$$\Rightarrow \operatorname{Proj}_X(\operatorname{Sym}\mathcal{L}^\vee) = X$$

§ Root gerbes as stacky Proj

We can modify the $\mathbb{G}_m$-action in the previous example:

$\mathbb{G}_m \xrightarrow{\wedge r} \mathbb{G}_m \curvearrowright \mathbb{V}(\mathcal{L})$ gives action $x.t = x^r t$
which corresponds to putting $\mathcal{L}^\vee$ in deg r.



$$\Rightarrow X_{(L,r)} = \operatorname{Proj}(\underset{0}{\mathcal{O}} \oplus \underset{r}{\mathcal{L}^\vee} \oplus \underset{2r}{\mathcal{L}^{\vee 2}} \oplus \cdots) \text{ is a global quotient stack.}$$

§ Root stacks of divisors as stacky Proj

$$\begin{tikzcd}
{[\operatorname{Spec} k[u]/\mu_r] = [\mathbb{A}^1/\mu_r]} \arrow[r] \arrow[d] & {[\mathbb{A}^1/\mathbb{G}_m]} \arrow[d, "{(\wedge r,\, \wedge r)}"] \\
{\operatorname{Spec} k[t] = \mathbb{A}^1} \arrow[r] & {[\mathbb{A}^1/\mathbb{G}_m]}
\end{tikzcd}
\qquad
\begin{tikzcd}
u^r \\
t \arrow[u, mapsto]
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{0}{\mathcal{O}_X} \oplus \overset{\text{gen } u}{\underset{1}{\mathcal{O}_X}} \oplus \underset{2}{\mathcal{O}_X} \oplus \cdots \oplus \overset{\text{gen } u^r}{\underset{r}{\mathcal{O}_X}} \oplus \cdots & = \mathcal{O}_X[\underset{r}{t}, \underset{1}{u}]/(u^r - t) & u^r \\
\underset{0}{\mathcal{O}_X} \oplus \underset{1}{0} \oplus \underset{2}{0} \oplus \cdots \oplus \overset{\text{gen } t}{\underset{r}{\mathcal{O}_X}} \oplus 0 \cdots \oplus \underset{2r}{\mathcal{O}_X} \arrow[u] & = \mathcal{O}_X[\underset{r}{t}] \arrow[u] & t \arrow[u, mapsto]
\end{tikzcd}$$

In general: Given $(\mathcal{L}, s)$ a generalized effective Cartier divisor

$$R = \underset{0}{\mathcal{O}_X} \oplus \underset{1}{\mathcal{L}^\vee} \oplus \underset{2}{\mathcal{L}^\vee} \oplus \cdots \oplus \underset{r}{\mathcal{L}^\vee} \oplus \underset{r+1}{\mathcal{L}^{\vee 2}} \oplus \underset{r+2}{\mathcal{L}^{\vee 2}} \oplus \cdots \oplus \underset{2r}{\mathcal{L}^{\vee 2}} \oplus \underset{2r+1}{\mathcal{L}^{\vee 3}} \oplus \cdots$$

multiplication $\underset{a}{\mathcal{L}^\vee} \otimes \underset{b}{\mathcal{L}^\vee} \longrightarrow \mathcal{L}^{\vee 2} \xrightarrow{1 \otimes s^\vee} \mathcal{L}^\vee \otimes \mathcal{O}_X = \underset{a+b}{\mathcal{L}^\vee}$ (if $a+b \leq r$)

Fact: $X_{(L,s,r)} = \operatorname{Proj}_X R = [R \setminus V(R_+) / \mathbb{G}_m]$

§2 Destackification

Thm (Bergh '17) $\mathcal{X}$ smooth stack with finite tame abelian stabilizers. Then $\exists$ diagram of smooth stacks

$$\begin{tikzcd} \mathcal{X}_n \arrow[r, "f_n"] \arrow[d, "g"'] & \mathcal{X}_{n-1} \arrow[r, "f_{n-1}"] & \cdots \arrow[r] & \mathcal{X}_1 \arrow[r, "f_1"] & \mathcal{X}_0 = \mathcal{X} \\ X' & & & & \end{tikzcd}$$

such that:

(1) $f_1, f_2, \ldots, f_n$ either blow-up in smooth center or root stack in smooth divisor

(2) g is a root stack in an snc divisor $D = D_1 + \cdots + D_m$ (+ a gerbe if $\mathcal{X}$ not generically a scheme)

(3) X' is an algebraic space.

SLOGAN: All stacks with finite tame abelian stabilizers can be built up using root stacks up to blow-ups.