Algebraic stacks #1

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Algebraic stacks builds upon many topics such as:

- étale topology/topos theory
- alg. spaces
- groupoids, categories fibered in groupoids, ...
- 2-categories
- ∞-categories, higher topos theory

⋮

Could give a full course on any single of these topics but won't.

Will try to avoid too much foundational material

§ Schemes

Classically: affine variety / scheme $X \xhookrightarrow{I} \mathbb{A}^n \iff I \subset k[x_1, \dots, x_n]$ ideal

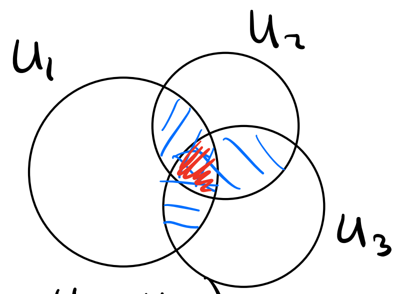
proj — " — $X \xhookrightarrow{I} \mathbb{P}^n \iff I \subset k[x_0, \dots, x_n]$ hom ideal up to saturation

Abstract varieties/schemes: Atlas $X = \bigcup_{i \in I} U_i \quad U_i = \operatorname{Spec} A_i$.

by gluing

- $\forall i$: $U_i = \operatorname{Spec} A_i$
- $\forall i, j$: $U_{ij} \subset U_i$ open, $\varphi_{ij} : U_{ij} \xrightarrow{\cong} U_{ji}$
- $\forall i, j, k$: $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $U_{ij} \cap U_{ik}$.

$\left(\varphi_{ii} = \mathrm{id},\ \varphi_{ij} = \varphi_{ji}^{-1},\ \varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk} \right)$



Cumbersome to describe maps etc.

Intrinsic definition 1: A scheme is:

- $X = (|X|, \mathcal{O}_X)$ locally ringed space

such that:

- $\exists\, X = \bigcup U_i$ open covering s.th. U_i affine scheme, i.e.

$|X| = \bigcup U_i$ open covering of top. spaces

$(|X|, \mathcal{O}_X)\big|_{U_i} \cong (\operatorname{Spec} A_i, \mathcal{O}_{\operatorname{Spec} A_i})$ isom. of loc ringed spaces.

Intrinsic definition 2: A scheme/alg space is a functor (today)

functor $\mathrm{Aff}^{op} \to \mathrm{Set}$

s.th ...

§ Moduli problems

Moduli problem = classify geom objects as points on a space/scheme/stack
the moduli space.

Ex: $\mathbb{P}^n = \{ L \subset k^{n+1} : \text{line through origin} \}$

$Gr(n,r) = \{ L \subset k^n : \text{linear subspace of dim } r \}$

$M_g = \{ C : \text{smooth proj conn. curve of genus } g \} / \sim$

Have only described moduli problem as a set but want a moduli space

§ Generalized points (Grothendieck)

Point $x \in X \iff$ map $\mathrm{Spec}\, k \xrightarrow{x} X$

Def Let T be a scheme. A T-point of X is a map

$$T \xrightarrow{x} X$$

$X(T) := \mathrm{Mor}(T, X)$ set of T-points.

Ex: $\mathbb{A}^1(T) = \mathrm{Mor}(T, \mathbb{A}^1) = \Gamma(T, \mathcal{O}_T)$

$\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T^\times) = \Gamma(T, \mathcal{O}_T)^\times$ $\qquad \mathbb{G}_m = \mathbb{A}^1 \setminus 0.$

$$\mathbb{P}^n(T) = \{ (\mathcal{L}, s_0, \dots, s_n) : \begin{array}{l} \mathcal{L} \text{ line bundle on } T \\ s_0, \dots, s_n \in \Gamma(T, \mathcal{L}) \text{ s.th.} \\ \forall t \in T : (s_0(t), \dots, s_n(t)) \neq 0 \end{array} \} / \sim$$

$$= \{ \mathcal{L} \in \mathrm{Pic}\, T, \ \mathcal{O}_T^{n+1} \overset{s}{\twoheadrightarrow} \mathcal{L} \} / \sim$$

$$\cong \{ \text{family of lines } (L_t \hookrightarrow \mathbb{A}^{n+1})_{t \in T} \}$$

$$:= \{ L \overset{i}{\hookrightarrow} \mathbb{A}^{n+1}_T : i \ \mathbb{G}_m\text{-equivariant} \}$$

$$\begin{tikzcd} L \arrow[rr, hook, "i"] \arrow[dr, "\text{l.b.}"'] & & \mathbb{A}^{n+1}_T \arrow[dl] \\ & T & \end{tikzcd}$$

$\mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{L}$ gives

$$\begin{tikzcd}
 & & \operatorname{Spec}_T(\operatorname{Sym}(\mathcal{O}_T^{n+1})) \arrow[d, phantom, "\parallel"] \\
L := \operatorname{Spec}_T(\operatorname{Sym}\mathcal{L}) \arrow[rr, hook] \arrow[dr, "\text{line bundle}"'] & & T \times \mathbb{A}^{n+1} \arrow[dl] \\
 & T &
\end{tikzcd}$$

Fiber over $t \in T$ is:
$$\begin{tikzcd}
\mathbb{A}^1 \cong L_t \arrow[rr] \arrow[dr] & & \mathbb{A}^{n+1} \arrow[dl] \\
 & t &
\end{tikzcd}$$

<u>Quotient pres</u>

$$\mathbb{P}^n = \mathbb{A}^{n+1} \setminus 0 / \mathbb{G}_m$$

$$\begin{tikzcd}
\mathbb{A}^{n+1} \setminus 0 \arrow[d, "\mathbb{G}_m\text{-torsor}"] \\
\mathbb{P}^n
\end{tikzcd}$$

<u>Ex</u>: (Moduli of closed subsch.) Let X be a proper scheme. <u>Hilbert functor</u>:

$$\operatorname{Hilb} X = \{ Z \overset{i}{\hookrightarrow} X : \text{closed subscheme} \}$$

$$(\operatorname{Hilb} X)(T) = \left\{ \begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "g"'] & X \times T \arrow[d, "p_2"] \\ & T \end{tikzcd} : \begin{array}{l} i \text{ closed subscheme} \\ g \text{ flat (and of finite pres.)} \end{array} \right\}$$

$$= \{ \text{family } (Z_t \hookrightarrow X)_{t \in T} \text{ of closed subscheme} \}$$

$$Z_t = g^{-1}(t)$$

X projective, ample line bundle $\mathcal{O}_X(1)$

$\Rightarrow$ Hilbert polynomial $\underset{\mathbb{Q}[t]}{\overset{\cap}{P_{\mathcal{O}_Z}(t)}} = \chi(X, \mathcal{O}_Z(t)) = \sum_i (-1)^i h^i(X, \mathcal{O}_Z(t))$
$= h^0(X, \mathcal{O}(t)) \quad \forall t \gg 0$

$$\operatorname{Hilb}^P X = \{ Z \hookrightarrow X : P_{\mathcal{O}_Z} = P \}$$

Functoriality: $T' \xrightarrow{f} T \quad \rightsquigarrow \quad f^*: X(T) \to X(T')$

$$g \longmapsto g \circ f$$

$$\begin{tikzcd} T' \arrow[r, "f"] \arrow[dr, "gf"'] & T \arrow[d, "g"] \\ & X \end{tikzcd}$$

Ex: $\mathbb{P}^n(T) \longrightarrow \mathbb{P}^n(T')$

$$\alpha : \mathcal{O}_T^{n+1} \twoheadrightarrow \mathcal{L} \longmapsto f^*\alpha : f^*\mathcal{O}_T^{n+1} \twoheadrightarrow f^*\mathcal{L}$$

$$\begin{tikzcd} L \arrow[r, hook, "i"] \arrow[dr] & \mathbb{A}^{n+1} \times T \arrow[d] \\ & T \end{tikzcd} \longmapsto \begin{tikzcd} L \times_T T' \arrow[r, hook, "f^*i"] \arrow[dr] & \mathbb{A}^{n+1} \times T' \arrow[d] \\ & T' \end{tikzcd}$$

Ex: $(\mathrm{Hilb}\, X)(T) \longrightarrow (\mathrm{Hilb}\, X)(T')$

$$(Z \hookrightarrow T) \longmapsto (Z \times_T T' \hookrightarrow T')$$

§ Functorial point of view

A scheme M is determined by its T-points

<u>Yoneda lemma</u>: Let $\mathcal{C}$ be a category. Then the functor

$$\begin{aligned} \mathcal{C} &\longrightarrow \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) \\ X &\longmapsto h_X = \mathrm{Mor}_{\mathcal{C}}(-, X) \end{aligned}$$

is fully faithful. (Yoneda embedding)

$$\begin{tikzcd} \mathrm{Sch} \arrow[r, "\text{f.f.}"] \arrow[dr, "\text{f.f.}"'] & \mathrm{Fun}(\mathrm{Sch}^{op}, \mathrm{Set}) \arrow[r, "\text{not f.f.}"] & \mathrm{Fun}(\mathrm{Aff}^{op}, \mathrm{Set}) \\ & \mathrm{Shv}(\mathrm{Sch}) \arrow[u, phantom, "\cup"] \arrow[r, "\cong"] & \mathrm{Shv}(\mathrm{Aff}) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

<u>Ex</u>: composition = restricted Yoneda also fully faithful.

<u>Functorial def of scheme</u>: A scheme is

- a functor $X: \mathrm{Aff}^{op} \longrightarrow \mathrm{Set}$

such that:

(a) X is a sheaf for the (big) Zariski topology.

(b) $\exists$ open covering $X = \bigcup U_i$ s.th. U_i affine scheme.

a bit awkward to define — for later

<u>Def</u>: A functor $X: \mathrm{Aff}^{op} \longrightarrow \mathrm{Set}$ is an <u>affine scheme</u> if it is <u>representable</u> i.e. in the image of the Yoneda embedding. That is:

$$\exists \, \mathrm{Spec}\, A \in \mathrm{Aff}: \quad X = h_{\mathrm{Spec}\, A} \quad \text{i.e.} \quad X(\mathrm{Spec}\, B) = \mathrm{Mor}(\mathrm{Spec}\, B, \mathrm{Spec}\, A) = \mathrm{Hom}(A, B)$$

<u>Ex</u>: $\mathbb{A}^1$ is the functor

$$\begin{aligned} \mathrm{Sch}^{op} &\longrightarrow \mathrm{Set} \\ T &\longmapsto \Gamma(T, \mathcal{O}_T) \end{aligned} \qquad \text{or} \qquad \begin{aligned} \mathrm{Aff}^{op} &\longrightarrow \mathrm{Set} \\ \mathrm{Spec}\, A &\longmapsto A \end{aligned}$$

Thm (Grothendieck 1960) If X is projective, then the functor

$$T \longmapsto (\mathrm{Hilb}\, X)(T) = \left\{ \begin{tikzcd} Z \arrow[r, hook, "\text{closed}"] \arrow[dr, "\text{flat}"'] & X \times T \arrow[d] \\ & T \end{tikzcd} \right\}$$

is a scheme, denoted $\mathrm{Hilb}\, X$. Moreover

$$\mathrm{Hilb}\, X = \coprod_{P \in \mathbb{Q}[t]} \mathrm{Hilb}^P X$$

and $\mathrm{Hilb}^P X$ projective scheme.

Proof quite involved. Idea:

(1) Reduce to $X = \mathbb{P}^n$ (easy)

(2) For fixed P, choose uniform big d s.t.

$$\mathrm{Hilb}^P X \hookrightarrow Gr(h^0(\mathbb{P}^n, \mathcal{O}(d)), P(d))$$

$$Z \longmapsto \underset{\substack{\| \\ k[x_0, \ldots, x_n]_d}}{\Gamma(\mathbb{P}^n, \mathcal{O}(d))} \twoheadrightarrow \underset{\substack{\| \\ (k[x_0, \ldots, x_n]/I)_d}}{\Gamma(\mathbb{P}^n, \mathcal{O}_Z(d))}$$

(or equiv. $I_d \subset k[x_0, \ldots, x_n]_d$)

Thm (Artin 1968) For any X proper, $\mathrm{Hilb}\, X$ is an algebraic space

Completely different proof (deformation thy + Artin approx)

$\rightsquigarrow$

$$\begin{tikzcd} Z \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Z_{univ} \arrow[d] \\ T \arrow[r, "{[Z]}", "f"'] & \mathrm{Hilb}\, X \end{tikzcd}$$

<u>Universal family:</u> $\mathrm{Hilb}\,X \xrightarrow{id} \mathrm{Hilb}\,X$

$$\Updownarrow$$

$$[id] \in (\mathrm{Hilb}\,X)(\mathrm{Hilb}\,X)$$

$$\begin{tikzcd} Z_{univ} \arrow[r, hook] \arrow[dr, "flat"'] & X \times \mathrm{Hilb}\,X \arrow[d] \\ & \mathrm{Hilb}\,X \end{tikzcd}$$

$$T \xrightarrow{f} \mathrm{Hilb}\,X \xrightarrow{id} \mathrm{Hilb}\,X$$

$$f$$

$$\Updownarrow$$

$$f^*([id]) = \left(\begin{tikzcd} Z \arrow[r, hook] \arrow[dr, "flat"'] & X \times T \arrow[d] \\ & T \end{tikzcd}\right) = [f] \in (\mathrm{Hilb}\,X)(Z)$$

$$\rightsquigarrow \begin{tikzcd} & X \times T \arrow[rr] \arrow[dl] & & X \times \mathrm{Hilb}\,X \arrow[dl] \\ Z \arrow[ur, hook] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & & Z_{univ} \arrow[ur, hook] \arrow[d] & \\ T \arrow[rr, "f"] & & \mathrm{Hilb}\,X & \end{tikzcd}$$

3 Moduli stacks

Ex (Moduli of curves)

$$M_g(T) = \text{"}\{\text{families } (C_t)_{t\in T} \text{ of smooth } \overset{\text{conn. proj}}{\text{genus } g} \text{ curves}\}\text{"}$$

$$= \left\{ C \xrightarrow{p} T : \begin{array}{l} \text{proper, flat such that} \\ \forall t:\ C_t := p^{-1}(t) \text{ smooth connected genus } g \end{array} \right\} \Big/ \sim$$

Not a scheme/alg space!

Reason: $\exists$ isotrivial families i.e. $C \longrightarrow T$ such that

(1) $C_t \cong C_{t'}$ $\forall t, t' \in T$

(2) $C \not\cong C_0 \times T$

If M_g was a scheme/alg space, then by (1) $C \to T$ would correspond to

$$\begin{tikzcd} T \arrow[r, "{[C]}"] & M_g \\ \operatorname{Spec}\mathbb{C} \arrow[u, "t"] \arrow[ur, "{[C_0]}"'] & \end{tikzcd} \quad \Rightarrow \text{ factor as } T \xrightarrow{f} \operatorname{Spec}\mathbb{C} \xrightarrow{[C_0]} M_g$$

$\Rightarrow C = f^* C_0 = C_0 \times T$ contradicts (2)

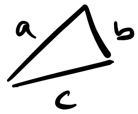
Ex: Pick your favorite elliptic curve $y^2 = f(x)$ (i.e. $y^2 z = x^3 + axz^2 + bz^3$)

Then $C = \{ty^2 = f(x)\} \hookrightarrow \mathbb{A}^1_t \times \mathbb{P}^2_{x,y,z}$

$$\begin{tikzcd} C = \{ty^2 = f(x)\} \arrow[d] \\ \mathbb{A}^1 = \operatorname{Spec}(\mathbb{C}[t]) \end{tikzcd}$$

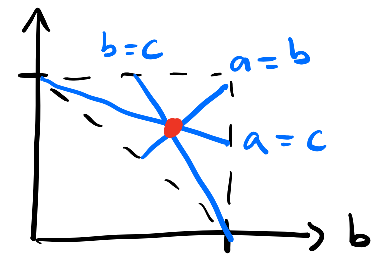
isotrivial family of elliptic curves over $\mathbb{A}^1$. Not trivial.

$\forall t \in \mathbb{C}, \exists \sqrt{t}$ $\quad \{(\sqrt{t}\,y)^2 = f(x)\} \cong \{y^2 = f(x)\}$

Ex (triangles) $\mathcal{M}_\Delta^{ord} = \Big\{$  $\Big\} = \left\{ (a,b,c) \in \mathbb{R}^3_{>0} : \begin{array}{l} a+b>c \\ b+c>a \\ c+a>b \end{array} \right\}$

Fix $a+b+c=2$
$c = 2-a-b$

$\Rightarrow 0 < a,b < 1$
$a+b > 1$

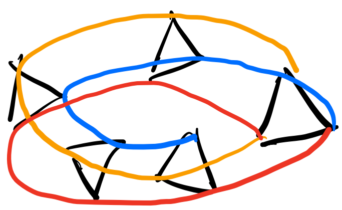


● = isosceles
● = equilateral

$\mathcal{M}_\Delta = \{$ triangles w/o ordered sides $\} = \mathcal{M}_\Delta^{ord} / S_3$

Objects w/ automorphism: $\mathrm{Aut}(\text{equilateral}) = S_3$
$\mathrm{Aut}(\text{isosceles}) = \mathbb{Z}/2\mathbb{Z}$

have iso-trivial families:



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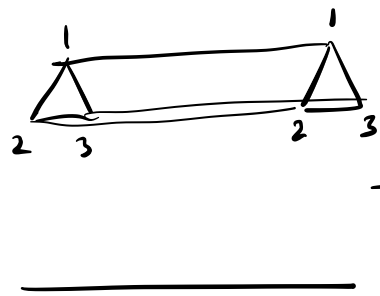
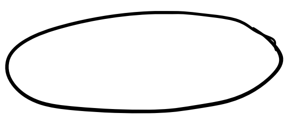


+ glue 1 – 2
2 – 3
3 – 1



S^1

Need to <u>remember</u> <u>automorphisms</u>!

§ Algebraic stacks

Solution to problem w/ isotrivial families: remember automorphisms

<u>Def</u>: An <u>algebraic stack</u> is

- a functor $\mathcal{X}: \mathrm{Aff}^{op} \longrightarrow \mathrm{Grpd}$

such that

(a) $\mathcal{X}$ is a sheaf (= stack) in étale topology.

(b) $\exists$ atlas $\coprod U_i = \operatorname{Spec} A_i \longrightarrow \mathcal{X}$ smooth surjective

and

(c) $\coprod U_i \longrightarrow \mathcal{X}$ represented by algebraic spaces

$\Leftrightarrow$ (c') $\Delta: \mathcal{X} \longrightarrow \mathcal{X} \times \mathcal{X}$ ——— " ———

Olsson's book takes (a) + (b) + (c') as definition.