Homework problems #2

To be handed in by email on Apr 17, 2024.

Problem 2.1 (Olsson, Exercise 4.G(a))**.** Let S be a scheme and let G be a sheaf of groups on the big étale site $\mathrm{Shv}(\mathbf{Sch}_S)$. Let BG be the fibered category of pairs (T, P) where $T \longrightarrow S$ is an S-scheme and $P \in \mathrm{Shv}(\mathbf{Sch}_T)$ is a torsor under $G|_{\mathbf{Sch}_T}$. A morphism $(T', P') \longrightarrow (T, P)$ in BG is a morphism of S-schemes $f\colon T' \longrightarrow T$ and an isomorphism $f^\flat\colon P' \longrightarrow P|_{\mathbf{Sch}_{T'}}$ of $G|_{\mathbf{Sch}_{T'}}$-torsors. Show that BG is a stack. *Hint: You don't have to prove that BG is a stack from scratch but can use the descent results that are already proven in Chapter 4.*

Problem 2.2 (Olsson, Exercise 5.D: part of the proof of Proposition 5.2.5 (ii) that we skipped on Lecture 6)**.** Let C be a site, and let $R \rightrightarrows X$ be an equivalence relation of sheaves. Let $p\colon X \longrightarrow X/R$ be the coequalizer in the category of sheaves. Show that the diagram

$$\begin{array}{ccc} R & \xrightarrow{s} & X \\ \downarrow{\scriptstyle t} & \square & \downarrow{\scriptstyle p} \\ X & \xrightarrow{p} & X/R \end{array}$$

is cartesian.

Hint: First show that the coequalizer of an equivalence relation of sheaves in the category of presheaves is a separated presheaf.

Problem 2.3. In Lecture 6, the following examples of non-separated algebraic spaces that are not schemes were mentioned (see lecture notes for details)

(1) (non-separated line)$/\mathbb{Z}/2\mathbb{Z}$ (complex conjugation action),

(2) (non-separated cross)$/\mathbb{Z}/2\mathbb{Z}$,

(3) (ramified covering of line)$/\mathbb{Z}/2\mathbb{Z}$,

(4) $\mathbb{A}^1_{\mathbb{C}}/\underline{\mathbb{C}}$, and

(5) $\operatorname{Spec}\overline{\mathbb{Q}}/\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Determine for each example whether it is locally separated and/or quasi-separated or neither. *Hint: It can be easier to work with an étale presentation that is affine rather than the given non-separated presentations in (a)–(c).*