David Rydh Algebraic stacks, Spring 2024

Homework problems #1

To be handed in by email on Apr 1, 2024.

Problem 1.1 (Alper, Exc 0.3.27). —

(a) Let $0 \leq r \leq n$ be integers and let $F\colon \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be the functor:

$$T \longmapsto \bigl\{q\colon \mathcal{O}_T^n \twoheadrightarrow \mathcal{O}_T^r \text{ surjective } \mathcal{O}_T\text{-homomorphism}\bigr\}/\sim$$

where two quotients are considered to be equal if their kernels are equal, i.e., $q \sim \varphi \circ q$ for all $\varphi \in \mathrm{GL}_r(\mathcal{O}_T)$. Is F represented by a scheme?

(b) Let $\mathbb{G}_m$ act on $\mathbb{A}^n \smallsetminus 0$ by $\lambda.(x_1, \dots, x_n) = (\lambda x_1, \dots, \lambda x_n)$. Give an explicit simple description of the presheaf quotient $G\colon \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$:

$$T \longmapsto (\mathbb{A}^n \smallsetminus 0)(T)/\mathbb{G}_m(T).$$

Is G a sheaf for the Zariski topology?

Problem 1.2. Let S be a noetherian scheme and let $f\colon X \longrightarrow Y$ be a morphism of S-schemes. Let $F\colon \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ be the functor:

$$T \longmapsto \{g\colon T \longrightarrow S \;:\; f_T\colon X \times_S T \longrightarrow Y \times_S T \text{ is an isomorphism}\}.$$

This is naturally a subfunctor of $h_S = \mathrm{Mor}(-, S)$.

(a) Show that if $X \longrightarrow S$ is flat and proper and $Y \longrightarrow S$ is proper, then $F \longrightarrow h_S$ is represented by an open immersion. *Hint: The fiberwise criterion of flatness could be useful, see Stacks project Tag 039A.*

(b) Find counter-examples to (a) when (i) $X \longrightarrow S$ is not flat, (ii) $X \longrightarrow S$ is not proper, or (iii) $Y \longrightarrow S$ is not proper. *Hint: The counter-examples are very simple and $S = \mathbb{A}^1$ suffices.*

Problem 1.3. Let G be a group and let BG denote the groupoid which has one object with $\mathrm{Aut}(*) = G$.

(a) Given two groups G, H, describe the groupoid $\mathrm{Hom}(BG, BH)$. What is its connected components?

(b) Show that the following diagram

$$\begin{array}{ccc} G & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & BG \end{array}$$

is naturally 2-cartesian (this involves specifying a 2-morphism).

(c) If H is a normal subgroup of G, show that

$$\begin{array}{ccc} BH & \longrightarrow & BG \\ \downarrow & & \downarrow \\ * & \longrightarrow & B(G/H) \end{array}$$

is 2-cartesian.